

THE CLOSED FORM SYNTHESIS OF SPRING PARAMETERS
TO BALANCE GIVEN FORCE FUNCTIONS IN PLANAR MECHANISMS

By

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to Keilah

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Major Department: Mechanical Engineering

The potential energy storage capabilities of linear springs are integrated with the non-linear motion of mechanisms to provide accurate approximation of desired counter-loading functions. Established techniques of kinematics are modified for the generation of both the analytical techniques and design procedures applied to springs. Many machine processes are dominated in their operating characteristics by large internal load variations. Reduction of such variations must enhance overall machine performance, yielding improved relative phasing and the ability to use smaller primary drive units.

A major change in perspective for mechanism design is accomplished by the ability to assume that the mechanism pre-exists. Heretofore it has always been necessary to deal with mechanisms on a positional basis where higher order motion could not be contemplated without also including the location of the mechanism. The set of analytics applied to springs is not affected by ignoring positional information. In fact, any combination of order properties, whether sequential or not, may be satisfied for the spring-related function so long as the kinematic constraint system pre-exists.

Proper employment of optimization concepts are discussed relative to analytical techniques. Several examples of spring synthesis are thoroughly examined and recommendations given for future investigations. Finally, an interactive computer program is shown which will handle all developed analytics. The program is shown to also be immediately applicable to the partial time state synthesis and classical kinematic synthesis of mechanisms.

A handwritten signature in cursive script, reading "Melvin P. Tsao".

Chairman

CHAPTER I

INTRODUCTION

Purpose

Numerous machine processes presently exist which are dominated in their operating characteristics by large load variations resulting from, but not limited to, d'Alembert forces, external work functions, and spring and damping elements. These variations are often the prime contributors to such phenomena as backlash, shaft wind-up, vibrations, and eventual fatigue failure. Reduction of such loading characteristics can only enhance machine operations, particularly precision of operation, and is especially desirable if implementable on an elementary and unified basis. Smoother load demands on primary drive systems further imply that such drives may be built smaller and more efficient, yielding both immediate and long-term savings. It is the purpose of this work to utilize springs to accomplish these goals through the development of a concise analytic formulation. Chapters II, III, and IV are devoted to this development. The mathematical concepts are integrated into useful form by a set of design procedures, found in Chapter V, and validated by several examples in Chapter VI.

Background

Historically, the study of springs in conjunction with unusual functional requirements may be divided into five categories as described by Conley [1]:

1. balancing of non-linear static forces,
2. dynamic force balancing for a constant energy level,

3. energy storage,
4. vibration resonance control or isolation, and
5. flexible link mechanisms.

In addition to dealing with the above list, this work will add the ability to directly produce a prespecified torque acting at a desired rotary input.

All real machines are comprised of four basic elements:

1. a kinetic energy storage capacity usually in terms of a mass or inertia,
2. a potential energy storage device, usually represented by a spring,
3. an energy removal device, due to viscous or coulomb friction, and
4. an external work function, such as an applied torque or force.

The conceptual model formulation for such devices has been thoroughly generalized by Benedict and Tesar [2] for both single- and multi-degree of freedom systems. It should be noted, however, that this modeling process is intended for the analysis of existing devices and does not explicitly point the direction for the design or implementation of the real elements.

Most devices are best understood if their behavior is linear in nature, and the spring is no exception. Even if the non-linear nature exhibited by some springs were to be well understood, the design of such springs built to accomplish specific non-linear tasks will often prove to be an insurmountable problem. Rauh [3] details an extensive set of such possible spring elements, but it may be readily seen that the analysis of these elements is not easy, while the design for even simple tasks will prove to be quite difficult. Thus the goal becomes one of "converting"

a relatively simple device into one which can perform highly unusual functional tasks. Two basic mechanical devices exist which can accomplish such conversions:

1. the cam system, and
2. the linkage.

Tesar and Matthew [4] note that a cam may be thought of as "an infinitely variable information storage device." Cams may be readily used to accomplish the goals alluded to earlier and this has, in fact, been previously reported in the above-cited reference, showing quite satisfactory results. The usefulness of cams for such purposes is, however, somewhat limited, due to the initial expense of creating the cam, and to the inability of the cam to carry large loads at high speeds.

Many efforts in the past have dealt with the question of spring utilization with linkages. As early as 1949, Quinn [5] investigated the problem of spring-loaded switch mechanisms, recognizing the value of the "partitioned" concept of energy. The treatment was one of dynamic response analysis and was somewhat imprecise -- due primarily to the lack of modern computational tools useful in solving the complex set of differential equations. Numerous researchers have directly involved themselves in attempting to "design" spring-augmented linkages to solve the circuit breaker problem, among them Barkan [6], who used a technique of piecewise linearization of the controlling response equations; Van Sickle and Goodman [7], who recognized force modeling concepts; and Genova [8], who utilized some elementary concepts for total system modeling. Genova also extensively studied methods to find equilibrium positions and corresponding phase plane plots of the system response. The majority of what she calls "synthesis" is, however, a classical optimization technique.

The same is true for work done by Kraniakas and Maunder [9] and by Seyffarth [10]. The preceding works [5-10] invariably deal with the question of dynamic response -- i.e., the "design" is of the "trial and error" variety. This is true even where some formal optimization scheme has been utilized to direct the search. None of the currently accepted concepts of synthesis, or parameter reduction by a direct analytical formulation, were used.

Addition of springs to correct static unbalance has also been examined by several researchers. Harmening [11] used a torsion spring at the input crank to counterbalance a downstream system mass. Utilizing a cumbersome technique for computation of linkage positions, he arrived at excellent results. Due to the location of the spring, however, the technique cannot be generalized to problems involving continuous rotation of the input. Haupt and Grewolls [12] investigated much the same problem and noted that their process was for single-position design, resulting in a set of solutions which may be searched to yield good results over a range of the independent rotation. Their report did not, however, actually conduct such a search. Livermore [13] looked at a planar representation of automotive suspension systems. A large-based computational capability led to the attempted solution of simultaneous, coupled differential equations in search of equilibrium positions. This effort likely represents the pinnacle of analysis for complex spring systems.

Much of the German literature is directed, not towards analytical solutions, but rather towards the conceptual implementation of springs. The field is dominated by Hain [14], who presents a detailed survey of other German publications on the subject. He also exhaustively documents the use of thin steel bands which are wrapped about specially curved surfaces to produce the required energy storage [15, 16, 17]. The major

difficulty appears to be the actual creation of such surfaces, a problem basically similar to that of cam geometry. Numerous "inversions" of the banding technique are shown and a kind of "type synthesis" is evolved. Again, very little effort is devoted to analytic formulations. Hain also uses a four-bar in conjunction with a torsion spring to create the desired non-linear effects [18]. The technique basically involves the utilization of a required energy curve together with the spring rate and preload to generate the motion specification for the linkage. Point position reduction techniques are then graphically applied to find solution linkages. The overall concept is similar to that utilized by Conley [1], although Hain was concerned only with finitely separated positions (FSP), while Conley was involved with analytical solutions for multiply separated positions (MSP).*

The design of flexible link mechanisms has witnessed relatively minor attention. Burns and Crossley [19] show the excellent replacement of a torsion spring-kinematic link by a cantilever which effectively pivots about a point located at one-sixth of the length of the link from the actual ground point. They have also examined the problem of utilizing flexible links to remove the difficulties normally associated with dead center positions [20].

Opposed to design, the dynamic response of flexible linkages has seen much recent attention, but the goals are outside the realm of the present effort; that is, the deformation problem usually encountered in these works is one of:

1. analysis -- "what path does the mechanism actually generate?"

*The concepts of FSP and MSP will be detailed later in the chapter.

2. kinematic synthesis -- "can one use known deformation properties in advance, thereby synthesizing a mechanism which will follow the desired path at a given operating speed?" and
3. vibration investigation -- "is the dynamic response likely to be dominated by coupling-induced transients?"

The last problem is actually quite closely associated with the first, except for the relatively higher speeds involved.

Actual application of synthesis techniques to improve cyclic dynamic response characteristics by means of springs has not received much attention. On the other hand, some previous research has achieved noteworthy results. Genova [21] attempts to minimize the torque fluctuations at the input for a single slider-crank with attached flywheel. She shows that by proper selection of parameters, the flywheel inertia may be reduced by 65% while maintaining the original degree of speed variations. This is accomplished by attaching a linear spring between a point on the input crank and the frame. Unfortunately, the technique was still a basic optimization approach, dealing with multiple Lagrangian-derived differential equations. One of the major analytic efforts to date was that of Lakshminarayana and Dizioglu [22], who tried to force a mechanism to retain the same natural frequency about the equilibrium position, regardless of amplitude. Their results were quite encouraging, using up to fourth order approximations for the desired functional relationships. It should be noted, however, that while the authors were dealing with spring-restrained mechanisms, the actual process was one of synthesis of mass content.

Of those few techniques previously noted which actually use a design procedure which may properly be called analytical synthesis, the method

used is the utilization of spring parameters and some functional relationship, usually energy or its derivatives, to generate a kinematic motion specification set for the creation of a linkage. Thus the pitfalls have always been the same as for other kinematic problems:

1. Solutions are not guaranteed to exist.
2. When solutions exist, there is no guarantee that they will satisfy the motion without a change of closure. This generally implies that the mechanism must be physically disassembled and then reassembled in the alternative closure.
3. Even when the above criteria are satisfied, the mechanism may not pass through the given positions in proper sequence.

The technique to be presented herein avoids the last two problems since it can assume that the mechanism pre-exists. Thus there still may be cases where no solution is found, but on the other hand, if a solution is found, it is guaranteed to be a "good" solution in the sense of criteria (2) and (3) above.

Function Approximations

Before proceeding to the analytic statement, it will be helpful to review some of the concepts and techniques available from kinematic synthesis.

A useful insight into the actual implementation of springs in mechanical systems may be gained by considering the general behavior of approximating functions. The limited number of parameters available in the kinematic problem means that any desired general function may only be approximated. Such approximation is generally accomplished by requiring that the function be identically satisfied at a finite number of precision points. Further, the analytic formulation which transforms the

required function into the geometric properties of link lengths or pin joint locations is precisely the concept of synthesis. When the function approximation is accomplished at the precision points without specifying higher order properties such as slope, curvature, etc., the approximation is said to be one of finitely separated positions, or FSP, as illustrated in Fig. I-1a. When the higher order properties are included at any one finitely separated position, the approximation is said to be one of infinitesimally separated positions, or ISP, as illustrated in Fig. I-1b. Finally, the properties of FSP and ISP may be combined to produce multiply separated positions, or MSP, shown in Fig. I-1c. For the kinematic problem, a unified approach exists [23] which may be directly applied to all the various combinations of MSP. Labeling finitely separated positions as P-P-...-P, and infinitesimally separated positions as PP...P, the possible combinations for 5 MSP are recognized as

P-P-P-P-P
 PP-P-P-P
 PP-PP-P
 PPP-P-P
 PPP-PP
 PPPP-P
 PPPPP.

The behavior of the approximating function relative to the desired function can be categorized according to the order of approximation. For various cases of MSP, such behavior is illustrated in Fig. I-2. In addition, it is possible that the desired function is satisfied for some intermediate point as seen in Fig. I-3, but the ability to predict such intersections is not within the realm of the synthesis process; i.e., the existence of such additional "precision points" may only be found by a subsequent analysis. The use of the term "precision points" is normally

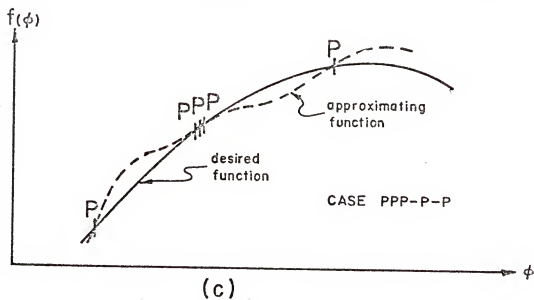
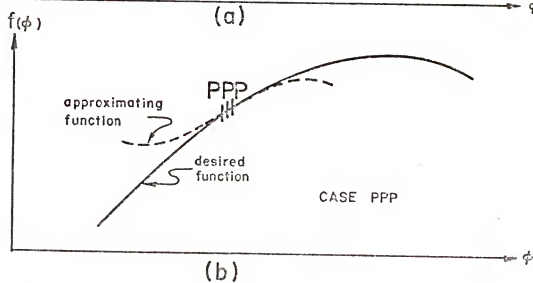
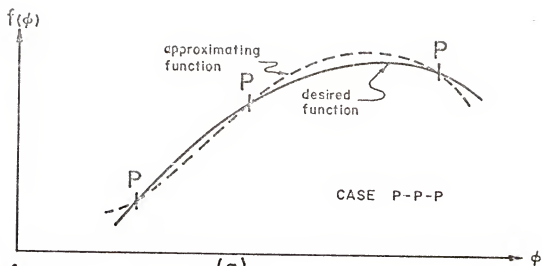


FIGURE I - I

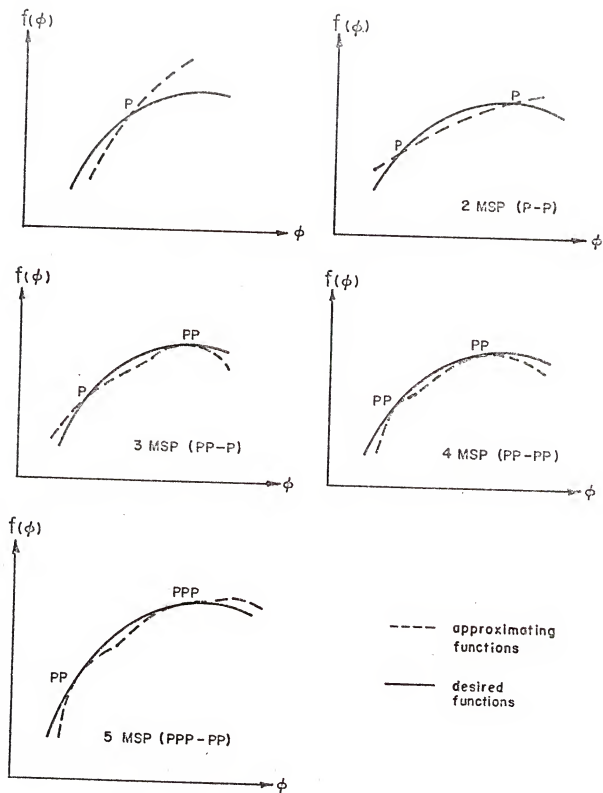


FIGURE I - 2

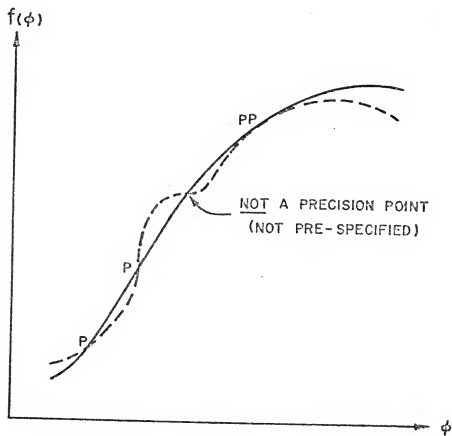


FIGURE I - 3

used to denote those MSP which are pre-specified. Hence the phrase is not applicable to the intermediate type of point currently under discussion.

The Tools of Synthesis

From planar kinematics the four tools of synthesis are recognized as:

1. the curvature transform,
2. angular inversions,
3. path cognates, or Roberts' Cognate mechanisms, and
4. angular cognates.

All of the above are applicable to the development of spring synthesis.

The last tool, that of angular cognates, is not as widely utilized in the field of kinematics as the others. This is, perhaps, unfortunate in that such cognates indeed represent a powerful tool in such processes as input-output function generation. Indeed, angular cognates, together with the curvature transform, form the basis for the work in springs by Conley [1].

The curvature transform itself is simply a technique in algebraic or vector notation for finding the center of the unique arc through three points. When the three points represent three positions of a point attached to a relatively moving plane, the center point is then available as the fixed pivot for the constraint link joining the relatively moving planes. Actually, when speaking of "the curvature transform", one is more likely to also be encompassing the total body of kinematic analytics. This body is usually more concerned with finding those points in a body which lie on a circular arc for more than three positions. A set of analytics quite similar to the curvature transform may also be developed for springs. It is the creation of such analytics which is the core of this work.

The idea of path cognates finds most of its usefulness in the interchange of the physical forms of solutions, while transferring the kinematic or time state to a new system element; for example, maintaining the parametrically generated shape of a point path, but passing the independent parameter to the input. The same principles again may be applied to the spring problem and are detailed in Chapter V. It should be noted, however, that the recent development by Myklebust and Tesar [24] for the specification of the partial time state of mechanisms has made the consideration of Roberts' Principle redundant for spring problems.

Finally, angular inversion is a technique of interchanging roles of the relatively moving planes and the constraint links. The inversion technique again takes a part in the spring problem, this time as a useful means for creating mechanism-spring combinations of different forms from those assumed by the analytics. Implementation of this technique is described in Chapter V and an example is given in Chapter VI.

Synthesis vs. Optimization

Returning to Fig. I-2, it may be noted that, for various solution choices, errors of relatively opposite behavior may be generated between precision points. Because kinematics is dominated by rigid body considerations, the use of multiple solutions is not feasible because the system would then be overconstrained. This restriction does not apply to deformable elements such as springs; i.e., the error effects can be used to cancel one another. The lack of precise analytical tools to point the way does not obviate the desirability of such implementation, but rather, demands efficient optimization techniques which in turn imply efficient analysis procedures. Such analysis tools have, in fact, been developed

by Curtis [25] and are currently being coded into generalized computer programs by Pollock [26].

As opposed to the optimization processes employed by earlier researchers, the heart of the matter in this instance is that the optimization is being conducted over a set which is already known to satisfy certain precision conditions. Furthermore, the researcher aligned with methods of classical optimization is not likely to recognize that such concurrent utilization of more than one result is liable to dramatically improve the system behavior. To illustrate further, suppose that the objective function of some optimization scheme may be represented by the contour chart of Fig. I-4. While for this particular case, it should be relatively easy to find the global minimum, it must be noted in all fairness that most optimization problems are actually not of the class represented by Fig. I-4, but rather, are far more complex. Suppose the researcher finds a number of local minima using the standard "tools of the trade." What, then, is the likelihood of recognizing that the pairwise (or triple-wise, etc.) application of the solutions yields a better result? Note that using two spring solutions implies the simultaneous optimization of two independently specified problems; i.e., both must independently reach a minimum simultaneously. The conclusion is obvious -- optimization is certainly a viable process, but the relatively simple and direct process of analytical synthesis, or parameter reduction, must always take precedence.

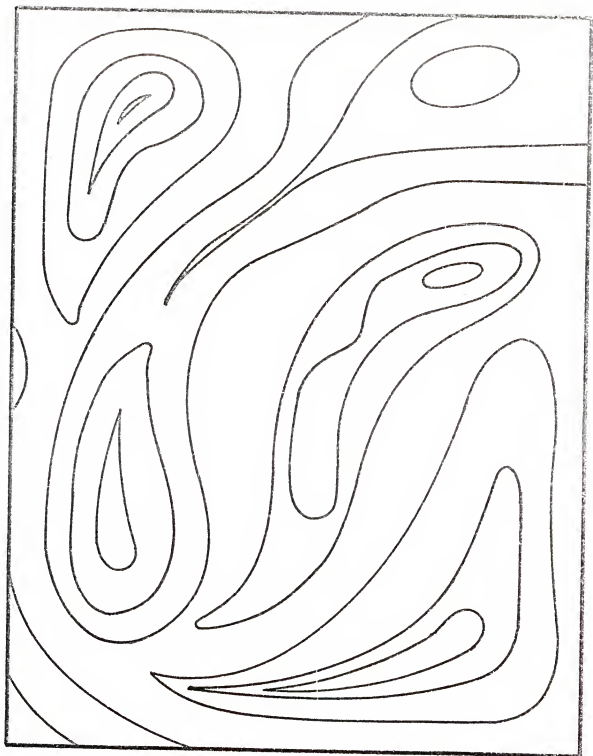


FIGURE I - 4

CHAPTER II

ANALYTIC FORMULATION FOR INSERTION OF A KNOWN SPRING CONSTANT INTO COPLANAR MOTION

The Controlling Energy Equation

The general energy function which may be generated by a linear spring is

$$\bar{E} = \frac{1}{2} KL^2.$$

An approximation for any such function may be accomplished by requiring that the above equation be satisfied for a finite number of precision conditions

$$\bar{E}_\ell = \frac{1}{2} KL_\ell^2, \quad \ell=0,1,2,\dots$$

where ℓ is the position counter for some independent variable. It will be briefly assumed that the positions are finitely separated. The spring is considered to be attached at $S(u,v)$ in the moving plane, E , and at $O_S(U_S, V_S)$ in the fixed reference, Σ , as shown in Figure II-1. Then the line segment L_ℓ is given by

$$L_\ell^2 = (U_\ell - U_S)^2 + (V_\ell - V_S)^2. \quad (2-1)$$

The coordinate transformation between the moving plane and the fixed reference, whose relative position is specified by parameters α_ℓ , β_ℓ , γ_ℓ , is well known to be

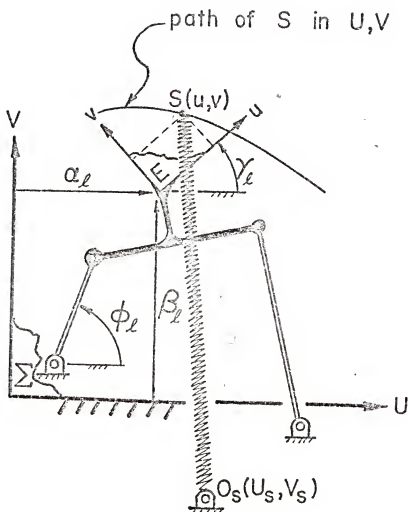


FIGURE II - I

$$\begin{aligned} U_\ell &= u \cos \gamma_\ell - v \sin \gamma_\ell + \alpha_\ell \\ V_\ell &= u \sin \gamma_\ell + v \cos \gamma_\ell + \beta_\ell. \end{aligned} \quad (2-2)$$

Rewriting the energy equation as

$$G_\ell = -E_\ell \delta + \frac{1}{2} L_\ell^2 = 0, \quad \delta = \frac{1}{K}$$

and by subtracting the reference position, $\ell = 0$, it becomes possible to form the difference

$$G_\ell - G_0 = -\delta(E_\ell - E_0) + \frac{1}{2}(L_\ell^2 - L_0^2) = 0. \quad (2-3)$$

Note that the kinematician asks the question: "Where is the point $A(u, v)$ in E which is moving on a circular arc about $O_g(U_g, V_g)$ in Σ ?" This question may be expressed as

$$L_\ell^2 - L_0^2 = 0.$$

The similarity of the above to Eq. (2-3) is readily apparent and suggests that the spring problem will closely follow the established kinematic formulations. Such is indeed the case and it is strongly felt that the concise statement of spring synthesis to follow is directly attributable to the algebraic-geometric approach to kinematics [23].

Equation (2-1) is now expanded by appropriate substitutions from Eq. (2-2) to yield

$$\begin{aligned} L_\ell^2 &= u^2 + v^2 + \alpha_\ell^2 + \beta_\ell^2 + 2(\alpha_\ell u \cos \gamma_\ell - \alpha_\ell v \sin \gamma_\ell + \beta_\ell u \sin \gamma_\ell \\ &\quad + \beta_\ell v \cos \gamma_\ell) - 2U_S(u \cos \gamma_\ell - v \sin \gamma_\ell + \alpha_\ell) - 2V_S(u \sin \gamma_\ell \\ &\quad + v \cos \gamma_\ell + \beta_\ell) + U_S^2 + V_S^2 \end{aligned}$$

and hence

$$\begin{aligned}
\frac{L_L^2 - L_0^2}{2} = & \left(\frac{\alpha_L^2 + \beta_L^2 - \alpha_0^2 - \beta_0^2}{2} \right) + u(\alpha_L \cos \gamma_L + \beta_L \sin \gamma_L \\
& - \alpha_0 \cos \gamma_0 - \beta_0 \sin \gamma_0) + v(-\alpha_L \sin \gamma_L + \beta_L \cos \gamma_L \\
& + \alpha_0 \sin \gamma_0 - \beta_0 \cos \gamma_0) - [uU_S + vV_S](\cos \gamma_L - \cos \gamma_0) \\
& + [vU_S - uV_S](\sin \gamma_L - \sin \gamma_0) - U_S(\alpha_L - \alpha_0) - V_S(\beta_L - \beta_0).
\end{aligned} \quad (2-4)$$

The parenthesized coefficients are recognized respectively as the zeroth derivative A_{mL} 's from kinematics ($m=0, \dots, 6$). Now, recognizing that energy stored in the spring results from energy removed from the mechanism, Eq. (2-4) may be written in the form

$$G_L - G_0 = \sum_{m=0}^7 Z_m A_{mL} = 0 \quad (2-5)$$

where $A_{7L} = \bar{E}_L - \bar{E}_0$, $Z_7 = \delta$. Derivatives of Eq. (2-5) may also be conceptually represented by Eq. (2-5) and the complete list of A_{mL} 's for spring synthesis is thus

$$\begin{aligned}
A_{0L} &= \frac{d^k}{d\gamma^k} \left\{ \frac{\alpha_L^2 + \beta_L^2 - \alpha_0^2 - \beta_0^2}{2} \right\} \\
A_{1L} &= \frac{d^k}{d\gamma^k} \{ \alpha_L \cos \gamma_L + \beta_L \sin \gamma_L - \alpha_0 \cos \gamma_0 - \beta_0 \sin \gamma_0 \} \\
A_{2L} &= \frac{d^k}{d\gamma^k} \{ -\alpha_L \sin \gamma_L + \beta_L \cos \gamma_L + \alpha_0 \sin \gamma_0 - \beta_0 \cos \gamma_0 \} \\
A_{3L} &= \frac{d^k}{d\gamma^k} \{ \cos \gamma_L - \cos \gamma_0 \} \\
A_{4L} &= \frac{d^k}{d\gamma^k} \{ \sin \gamma_L - \sin \gamma_0 \} \\
A_{5L} &= \frac{d^k}{d\gamma^k} \{ \alpha_L - \alpha_0 \} \\
A_{6L} &= \frac{d^k}{d\gamma^k} \{ \beta_L - \beta_0 \} \\
A_{7L} &= \frac{d^k}{d\gamma^k} \{ \bar{E}_L - \bar{E}_0 \}
\end{aligned} \quad (2-6)$$

which in turn requires

$$\begin{aligned}
 Z_0 &= 1 & Z_4 &= v\dot{U}_S - u\dot{V}_S \\
 Z_1 &= u & Z_5 &= -\dot{U}_S \\
 Z_2 &= v & Z_6 &= -\dot{V}_S \\
 Z_3 &= -(u\dot{U}_S + v\dot{V}_S) & Z_7 &= \delta
 \end{aligned} \tag{2-7}$$

The fact that the problem has been formulated for the difference between two positions is viable since by far the largest set of problems to be attacked is that of torque balancing. This in turn implies a derivative or a "differencing" of the energy curve. Mathematically, then, the absolute energy level is unimportant, although from a physical viewpoint, excessive storage of energy can lead to practical limitations. The ability to specify the energy reference level will be taken up in a later chapter.

Recognizing from Eq. (2-7) that

$$\begin{aligned}
 Z_3 &= Z_1 Z_5 + Z_2 Z_6 \\
 Z_4 &= Z_1 Z_6 - Z_2 Z_5,
 \end{aligned} \tag{2-8}$$

Eq. (2-5) is rewritten as

$$\begin{aligned}
 G_\ell - G_0 &= A_{0\ell} + Z_1 A_{1\ell} + Z_2 A_{2\ell} + (Z_1 Z_5 + Z_2 Z_6) A_{3\ell} \\
 &\quad + (Z_1 Z_6 - Z_2 Z_5) A_{4\ell} + Z_5 A_{5\ell} + Z_6 A_{6\ell} + Z_7 A_{7\ell} = 0.
 \end{aligned} \tag{2-9}$$

This is now the final form of the generalized constraint function to be satisfied for $\ell = 1, 2, \dots$

When the spring constant is known, implying Z_7 is known, and for designer-specified values for the $A_{m\ell}$'s, separation of Eq. (2-9) in terms of Z_5 and Z_6 gives

$$(A_{52} + Z_1 A_{12} + Z_2 A_{22} + Z_7 A_{72}) + Z_5 (Z_1 A_{32} - Z_2 A_{42} + A_{52}) \\ + Z_6 (Z_1 A_{42} + Z_2 A_{32} + A_{62}) = 0$$

which is written as

$$(\bar{D}_2 + \bar{G}_2) + Z_5 E_2 + Z_6 F_2 = 0, \quad \bar{G}_2 = Z_7 A_{72}. \quad (2-10)$$

Thus the terms $\bar{D}_2$, E_2 , F_2 retain precisely the same definitions [23] as for the pure kinematic problem. The question now to be addressed is the solution techniques of Eq. (2-10), based upon the possible choices of the position counter, 2 .

Two Multiply Separated Positions ($2 = 1$)

For $2 = 1$, Eq. (2-10) represents the straight line locus of Z_5 , Z_6 for chosen values of $\bar{D}_1$, E_1 , F_1 , $\bar{G}_1$. Written in terms of the physical coordinates, this becomes

$$E_1 U_S + F_1 V_S = \bar{D}_1 + \bar{G}_1 \quad (2-11)$$

where

$$\begin{aligned} \bar{D}_1 &= A_{01} + uA_{11} + vA_{21} \\ E_1 &= uA_{31} - vA_{41} + A_{51} \\ F_1 &= uA_{41} + vA_{31} + A_{61} \\ \bar{G}_1 &= \delta A_{71}. \end{aligned} \quad (2-12)$$

For $\bar{G}_1 = 0$, this is precisely the equation obtained for the kinematic constraint locus (or more compactly, KCL), and may be written as

$$V_S = -\frac{E_1}{F_1} U_S + \frac{\bar{D}_1}{F_1}$$

where the "g" subscript is used to denote geometric properties. The kinematic pole U_p, V_p or u_p, v_p , denoted by P_{01} is the point where no motion occurs from $\ell = 0$ to $\ell = 1$ even though coplanar motion exists. One may note that the kinematic pole represents a singularity in Eq. (2-11); i.e., E_1 and F_1 go to zero simultaneously and hence

$$u_p A_{31} - v_p A_{41} + A_{51} = 0$$

$$u_p A_{41} + v_p A_{31} + A_{61} = 0$$

so that

$$u_p = -(A_{31}A_{51} + A_{41}A_{61})/(A_{31}^2 + A_{41}^2)$$

$$v_p = (A_{41}A_{51} - A_{31}A_{61})/(A_{31}^2 + A_{41}^2).$$

It may be readily demonstrated that if $E_1 = F_1 = 0$, then $\bar{D}_1 = 0$ and Eq. (2-11) remains satisfied for the kinematic problem regardless of the values of U_g, V_g . This may in turn be interpreted as allowing a kinematic constraint link to be attached to the pole in the moving plane and any point in the fixed plane or, conversely, to be attached to the pole in the fixed plane and any point in the moving plane; i.e., point position reduction.

For the case where $\bar{C}_1 \neq 0$, the same singularity exists, but Eq. (2-11) cannot be satisfied for finite values of U_g, V_g . Thus the kinematic pole is not a viable solution for the spring problem. It will be shown in a later section that the consequences of this result are far-reaching. Equation (2-11) also shows that the spring constraint locus (SCL) differs from the KCL by a constant; i.e., the two lines are parallel. These properties are illustrated in Fig. II-2.

Solutions of elementary kinematic problems by graphical techniques have classically been of great interest to designers. For finitely

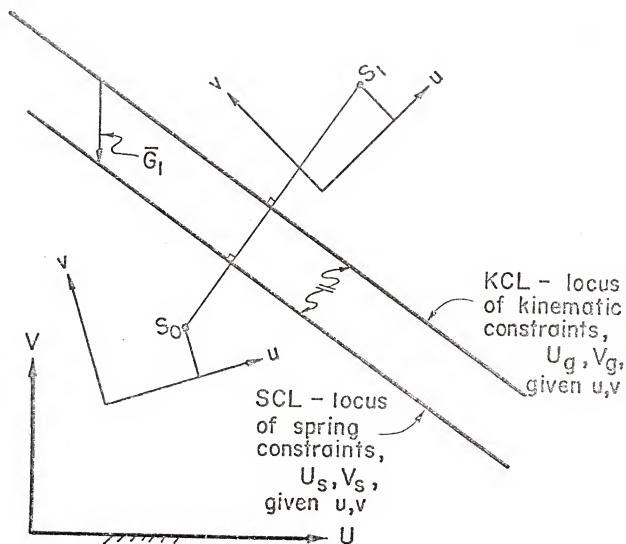


FIGURE II - 2

separated positions, a method for graphical solution of the spring problem will now be outlined. Referring to Fig. II-3a, note that two lines have been drawn, together with a circle. The line which passes through the center may be considered to have a length L_1 . The line tangent to the circle is denoted by L_0 . Hence the distance from the center of the circle is

$$L_1^2 - L_0^2 = 2\delta(\bar{E}_1 - \bar{E}_0),$$

assuming $\bar{E}_1 > \bar{E}_0$. A double infinity of such two-line combinations exists. Requiring that a circle of the above-described radius be drawn about $S(u,v)$ for position $\ell = 1$ and that an arc about any candidate $O_S(U,V)$ pass through

1. the intersection of the circle and the circle tangent through O_S , and
2. $S(u,v)$ for position $\ell = 0$,

reduces the solution set to a single infinity of choices for O_S . Further, it is known from the preceding analytic formulation that the SCL must be a straight line. The following procedure can thus be used to graphically locate the SCL:

1. Locate the chosen u,v in positions $\ell = 0,1$; i.e., S_0, S_1 .
2. At S_1 , draw a circle of radius $\sqrt{2\delta(\bar{E}_1 - \bar{E}_0)}$.
3. Choose any point, C , on the circle, not on S_0S_1 .
4. Draw the perpendicular bisector of S_0C .
5. Draw the tangent to the circle at C .
6. Locate point D as the intersection of steps 4 and 5.
7. The SCL is the line through D , perpendicular to S_0S_1 .

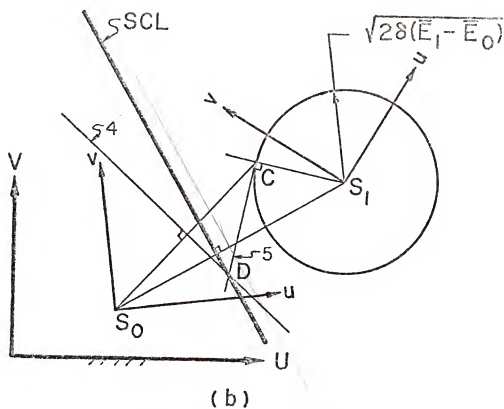
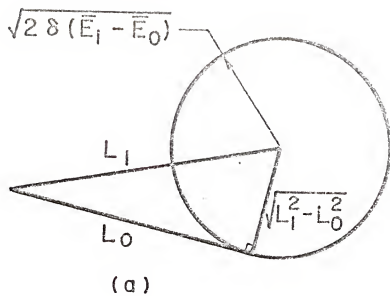


FIGURE II - 3

This procedure is illustrated in Fig. II-3b. If $\bar{E}_1 < \bar{E}_0$, the roles of $\ell = 0, 1$ may be interchanged; i.e., the circle is now drawn in the $\ell = 0$ position at S_0 .

Three Multiply Separated Positions ($\ell = 1, 2$)

For $\ell = 2$, it is imperative that all statements about position 2 relative to position 0 yield the same generalized results as obtained for position 1 relative to position 0; e.g., new results are obtained for P_{02} , KCL, SCL, etc. Positions 1 and 2 may also be combined to yield similar information, albeit not so readily since the original assumption was the subtraction of the zero (reference) position from subsequent positions.

It has been shown for $\ell = 1$ that straight lines represent the KCL and SCL. The same must be true for $\ell = 2$ and, furthermore, these lines must intersect, in general. In fact, so long as the kinematic solution is finite, so must the spring solution be finite. For finitely separated positions, the graphical process is shown in Fig. II-4 where the procedure outlined previously has been applied for positions $\ell = 0, 2$ as well as $\ell = 0, 1$.

The intersection of the KCL for $\ell = 0, 1$ and $\ell = 0, 2$ provides the kinematic fixed pivot, O_g , for the specified motion and choice of $A(u, v)$. The intersection of the SCL for $\ell = 0, 1$ and $\ell = 0, 2$ provides the spring fixed pivot, O_s , for the specified motion, choice of $S(u, v)$, and energy content. The overall process is shown in Fig. II-5.

In order to develop these ideas analytically, Eq. (2-10) is written for $\ell = 1, 2$ as

$$\begin{aligned} F_1 U_S + F_1 V &= \bar{D}_1 + \bar{G}_1 \\ F_2 U_S + F_2 V &= \bar{D}_2 + \bar{G}_2 \end{aligned} \quad (2-13)$$

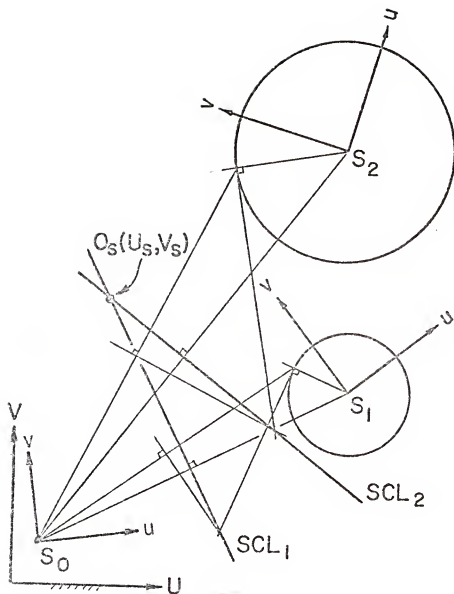


FIGURE II - 4

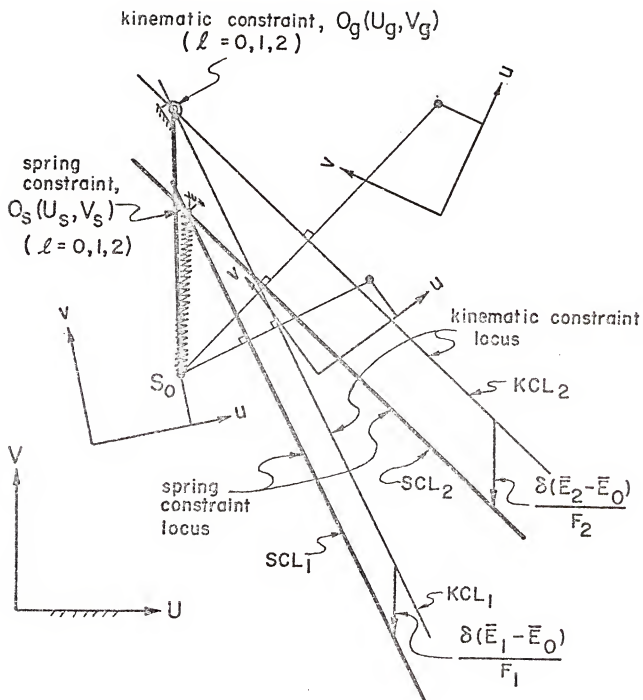


FIGURE II - 5

from which

$$\begin{aligned}
 U_s &= \frac{\begin{vmatrix} \bar{D}_1 & F_1 \\ \bar{D}_2 & F_2 \end{vmatrix}}{\Delta} + \frac{\begin{vmatrix} \bar{G}_1 & F_1 \\ \bar{G}_2 & F_2 \end{vmatrix}}{\Delta} \\
 V_s &= \frac{\begin{vmatrix} E_1 & \bar{D}_1 \\ E_2 & \bar{D}_2 \end{vmatrix}}{\Delta} + \frac{\begin{vmatrix} E_1 & \bar{G}_1 \\ E_2 & \bar{G}_2 \end{vmatrix}}{\Delta}
 \end{aligned}
 \tag{2-14}$$

where $\Delta = E_1 F_2 - E_2 F_1$.

Equation (2-14) has been partitioned in order to again separate the pure kinematic concepts; i.e., if $\bar{G}_1 = \bar{G}_2 = 0$, the second ratio disappears and the first ratio in each case locates the kinematic constraint, O_g . Thus Eq. (2-14) may be represented by

$$\begin{aligned}
 U_s &= U_g + (\bar{G}_1 F_2 - \bar{G}_2 F_1) / \Delta \\
 V_s &= V_g + (E_1 \bar{G}_2 - E_2 \bar{G}_1) / \Delta
 \end{aligned}
 \tag{2-15}$$

Equation (2-14) is referred to as the three-position transformation for $u, v \rightarrow U_s, V_s$, and is unique for $\Delta \neq 0$. $\Delta = 0$ implies a locus in u, v which, for general MSP, is referred to as the three-position circle locus. For 3 ISP, this circle becomes the inflection circle, meaning that the chosen u, v exhibits an inflection in its point path. This in turn implies that the corresponding fixed pivot for the kinematic constraint lies at infinity. For the spring problem, choices of u, v on the circle locus require that O_s lie at infinity. Such solutions have previously been shown to be untenable.

Note that it is indeed possible to attach a kinematic constraint and a spring to the same point u, v . This means that, if desired, a spring may be attached to a constraint link and the frame. Further, absolute

angular and length reference is then lost and more convenient locations for spring attachment may be used as illustrated in Fig. II-6.

Four Multiply Separated Positions ($l = 1, 2, 3$)

To solve the problem of four MSP, Eq. (2-10) is written for the three position conditions as

$$\bar{D}_1 + \bar{G}_1 - E_1 U_S - F_1 V_S = 0$$

$$\bar{D}_2 + \bar{G}_2 - E_2 U_S - F_2 V_S = 0$$

$$\bar{D}_3 + \bar{G}_3 - E_3 U_S - F_3 V_S = 0$$

which may in turn be represented by

$$\begin{bmatrix} (\bar{D}_1 + \bar{G}_1) & E_1 & F_1 \\ (\bar{D}_2 + \bar{G}_2) & E_2 & F_2 \\ (\bar{D}_3 + \bar{G}_3) & E_3 & F_3 \end{bmatrix} \begin{Bmatrix} -1 \\ U_S \\ V_S \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (2-16)$$

Equation (2-16) can be non-trivially satisfied only if

$$\begin{vmatrix} \bar{D}_1 & E_1 & F_1 \\ \bar{D}_2 & E_2 & F_2 \\ \bar{D}_3 & E_3 & F_3 \end{vmatrix} + \begin{vmatrix} \bar{G}_1 & E_1 & F_1 \\ \bar{G}_2 & E_2 & F_2 \\ \bar{G}_3 & E_3 & F_3 \end{vmatrix} = 0. \quad (2-17)$$

The first term of Eq. (2-17) is the kinematic "circle point curve"; i.e., a cubic equation of the form

$$(u^2 + v^2)(H_1 u + H_2 v) + H_3 u^2 + H_4 v^2 + H_5 uv + H_6 u + H_7 v + H_8 = 0. \quad (2-18)$$

Equation (2-18) may be interpreted as the locus of all points A(u,v) which are moving through four positions on a circular arc about respective points O_a , in the fixed reference.

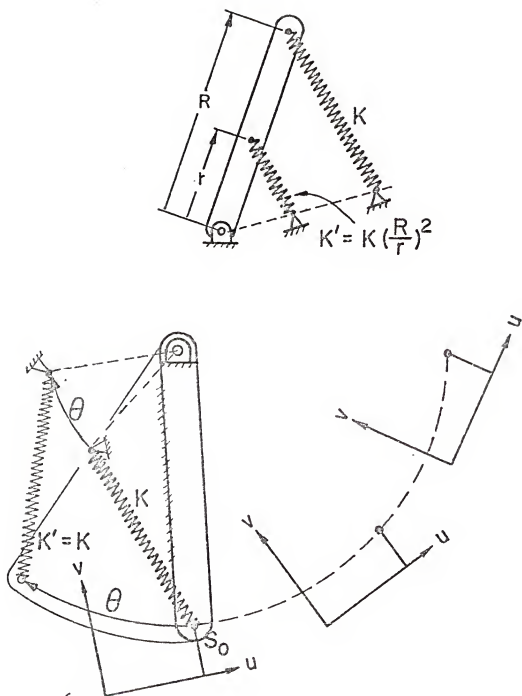


FIGURE II - 6

The second term of Eq. (2-17) is expanded to

$$\begin{vmatrix} \bar{G}_1 & (A_{51} + uA_{31} - vA_{41}) & (A_{61} + uA_{41} + vA_{31}) \\ \bar{G}_2 & (A_{52} + uA_{32} - vA_{42}) & (A_{62} + uA_{42} + vA_{32}) \\ \bar{G}_3 & (A_{53} + uA_{33} - vA_{43}) & (A_{63} + uA_{43} + vA_{33}) \end{vmatrix}$$

which can be further expanded to

$$\begin{vmatrix} \bar{G}_1 & A_{51} & A_{61} \\ \bar{G}_2 & A_{52} & A_{62} \\ \bar{G}_3 & A_{53} & A_{63} \end{vmatrix} + u \begin{vmatrix} \bar{G}_1 & A_{31} & A_{61} \\ \bar{G}_2 & A_{32} & A_{62} \\ \bar{G}_3 & A_{33} & A_{63} \end{vmatrix} + u \begin{vmatrix} \bar{G}_1 & A_{51} & A_{41} \\ \bar{G}_2 & A_{52} & A_{42} \\ \bar{G}_3 & A_{53} & A_{43} \end{vmatrix} + v \begin{vmatrix} \bar{G}_1 & -A_{41} & A_{61} \\ \bar{G}_2 & -A_{42} & A_{62} \\ \bar{G}_3 & -A_{43} & A_{63} \end{vmatrix} \\ + v \begin{vmatrix} \bar{G}_1 & A_{51} & A_{31} \\ \bar{G}_2 & A_{52} & A_{32} \\ \bar{G}_3 & A_{53} & A_{33} \end{vmatrix} + u^2 \begin{vmatrix} \bar{G}_1 & A_{31} & A_{41} \\ \bar{G}_2 & A_{32} & A_{42} \\ \bar{G}_3 & A_{33} & A_{43} \end{vmatrix} + v^2 \begin{vmatrix} \bar{G}_1 & -A_{41} & A_{31} \\ \bar{G}_2 & -A_{42} & A_{32} \\ \bar{G}_3 & -A_{43} & A_{33} \end{vmatrix} .$$

The coefficient of the uv term is zero and the coefficients of the u^2 and v^2 terms are identical, resulting in a circle which may be more compactly written as

$$(u^2 + v^2) |\bar{G}_\ell A_{3\ell} A_{4\ell}| + u \{ |\bar{G}_\ell A_{3\ell} A_{6\ell}| + |\bar{G}_\ell A_{5\ell} A_{4\ell}| \} \\ + v \{ |\bar{G}_\ell A_{6\ell} A_{4\ell}| + |\bar{G}_\ell A_{5\ell} A_{3\ell}| \} + |\bar{G}_\ell A_{5\ell} A_{6\ell}| = 0.$$

Note that δ may be factored out of each determinant; i.e., the circle is independent of the spring constant. This result is somewhat surprising and also of great importance as will be shown later in this section. Thus the overall form of Eq. (2-17) is that of a circular cubic which may be written as

$$(u^2 + v^2) (\underline{H}_1 u + \underline{H}_2 v) + \underline{H}_3 u^2 + \underline{H}_4 v^2 + \underline{H}_5 uv + \underline{H}_6 u + \underline{H}_7 v + \underline{H}_8 = 0 \quad (2-19)$$

where

$$\begin{aligned}
 \underline{H}_1 &= H_1 \\
 \underline{H}_2 &= H_2 \\
 \underline{H}_3 &= H_3 + |\bar{G}_\ell A_{3\ell} A_{4\ell}| \\
 \underline{H}_4 &= H_4 + |\bar{G}_\ell A_{3\ell} A_{4\ell}| \\
 \underline{H}_5 &= H_5 \\
 \underline{H}_6 &= H_6 + |\bar{G}_\ell A_{3\ell} A_{6\ell}| + |\bar{G}_\ell A_{5\ell} A_{4\ell}| \\
 \underline{H}_7 &= H_7 + |\bar{G}_\ell A_{6\ell} A_{4\ell}| + |\bar{G}_\ell A_{5\ell} A_{3\ell}| \\
 \underline{H}_8 &= H_8 + |\bar{G}_\ell A_{5\ell} A_{6\ell}|
 \end{aligned} \tag{2-20}$$

Since the slope of the real asymptote to such a cubic is given by

$$t = -\frac{\underline{H}_1}{\underline{H}_2} = -\frac{H_1}{H_2}, \tag{2-21}$$

it is apparent that the direction of the asymptote is the same for the kinematic and spring problem. Further assuming that the asymptote may be represented by $v = tu + q$, it may be shown that the value of q for the real asymptote is

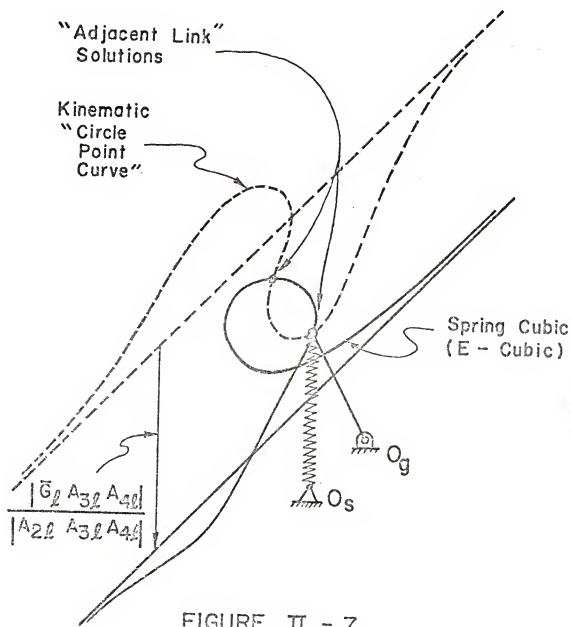
$$q = (\underline{H}_1 \underline{H}_2 \underline{H}_5 - \underline{H}_1^2 \underline{H}_4 - \underline{H}_2^2 \underline{H}_3) / \underline{H}_2 (\underline{H}_1^2 + \underline{H}_2^2) \tag{2-22}$$

and, using substitutions from Eq. (2-20)

$$q_s = q_g - \frac{|\bar{G}_\ell A_{3\ell} A_{4\ell}|}{\underline{H}_2} = q_g - \frac{|\bar{G}_\ell A_{3\ell} A_{4\ell}|}{|A_{2\ell} A_{3\ell} A_{4\ell}|} \tag{2-23}$$

These properties are illustrated in Fig. II-7. This cubic for attachment of the spring in the moving plane will be referred to as the E-cubic.

It is a well-known property of such curves that if a single solution u, v is known, a translation of coordinates to that point may be made,



followed by a parameterization of the curve in polar coordinates (r, θ) , which has as its form a quadratic in r . For the kinematic problem, it is known that the focus of the curve lies on the curve. Further, the focus may be found directly. Fortunately, the focus almost invariably lies in the "useful" part of the circular cubic. Even if the focus were not so convenient, it may be recognized that the pole, P_{01} , satisfies the kinematic cubic since $D_1 = E_1 = F_1 = 0$. The same, of course, is true for P_{02} and P_{12} .

Unfortunately, the spring problem does not yield such elegant results since:

1. the focus of the spring cubic does not lie on the curve,* and
2. the poles no longer trivially satisfy Eq. (2-16) for $G_k \neq 0$.

There is, however, yet another point on the cubic which may be easily located, that of the intersection of the cubic and its asymptote. There must be only one such intersection and it may be found by substituting $v = tu + q$ into Eq. (2-19) which then reduces to a first order equation in u . Having found u , substitution back into $v = tu + q$ yields the corresponding value of v . This process is detailed in Appendix A. The major drawback associated with this point is that it is often not in the "useful" area of the curve. A computing algorithm for correcting this deficiency is developed in Appendix B. Once the E-cubic is established from the quadratic form in terms of r, θ , one may select from the curve any $S(u, v)$. The coordinates are then returned to the three-position transformation (Eq. (2-14)) to determine the corresponding fixed pivot location, U_s, V_s . It should be noted that the locus of all $O_s(U_s, V_s)$ in

*It may be shown that the focus of the spring cubic coincides with the focus of the kinematic cubic.

reference plane Σ is also a circular cubic, which will be referred to as the Σ -cubic.

It was previously stated that the difference between the E-cubic and the kinematic cubic is a circle. Hence, points of intersection between the two cubics must lie on the circle. Further, it is known from the theory of algebraic curves that the intersection of a circle and a cubic will produce a sixth order polynomial, the roots of which correspond to the intersections. For these particular curves, however, two of the intersections must lie at the imaginary I, J points at infinity. The remaining maximum of four real intersections may be used to produce "adjacent link" solutions for four positions. An analytic method for finding these solutions will now be presented.

The kinematic circular cubic is

$$(u^2 + v^2)(H_1u + H_2v) + H_3u^2 + H_4v^2 + H_5uv + H_6u + H_7v + H_8 = 0.$$

The difference between the kinematic cubic and E-cubic may be written as

$$u^2 + v^2 + au + bv + c = 0. \quad (2-24)$$

Recall now the independence of Eq. (2-19) from δ which is at first somewhat surprising until one realizes that the desired end result is not only $S(u,v)$ for spring attachment, but also $A(u,v)$ for kinematic constraint, and that the kinematic link may be regarded as a spring where $K \rightarrow \infty$. Ultimately, the return to the three-position transformation must resolve the question of spring constant and the corresponding fixed pivot location.

Equation (2-24) is now rewritten as

$$(u + a')^2 + (v + b')^2 + c' = 0$$

which, expanded, becomes

$$u^2 + 2a'u + a'^2 + v^2 + 2b'v + b'^2 + c' = 0$$

where

$$a' = \frac{1}{2}a, \quad b' = \frac{1}{2}b, \quad c' = c - \frac{1}{4}(a^2 + b^2).$$

A transformation of coordinates to the center of the circle is now made:

$$\begin{aligned} x &= u + a' = u + \frac{1}{2}a \\ y &= v + b' = v + \frac{1}{2}b. \end{aligned} \quad (2-25)$$

Substitution of this transformation into the circular cubic yields

$$(x^2 + y^2)(\zeta_1 x + \zeta_2 y) + \zeta_3 x^2 + \zeta_4 y^2 + \zeta_5 xy + \zeta_6 x + \zeta_7 y + \zeta_8 = 0 \quad (2-26)$$

where

$$\zeta_1 = H_1$$

$$\zeta_2 = H_2$$

$$\zeta_3 = H_3 - (3H_1 a + H_2 b)/2$$

$$\zeta_4 = H_4 - (H_1 b + 3H_2 a)/2$$

$$\zeta_5 = H_5 - H_1 b - H_2 a$$

$$\zeta_6 = H_6 - H_3 a - H_5 b/2 + (3H_1 a^2 + 2H_2 ab + H_1 b^2)/4$$

$$\zeta_7 = H_7 - H_4 b - H_5 a/2 + (2H_1 ab + 3H_2 b^2 + H_2 a^2)/4$$

$$\zeta_8 = H_8 + (H_3 a + H_4 b^2 + H_5 ab - 2H_6 a - 2H_7 b)/4 + (a^2 + b^2)(H_1 a + H_2 b)/8.$$

Recognizing from previous development that

$$x^2 + y^2 + c' = 0,$$

appropriate substitutions may be made in Eq. (2-26) to give

$$(-c')(\zeta_1\sqrt{-c' - y^2} + \zeta_2 y^2) + \zeta_3(-c' - y^2) + \zeta_4 y^2 \\ + \zeta_5 y\sqrt{-c' - y^2} + \zeta_6\sqrt{-c' - y^2} + \zeta_7 y + \zeta_8 = 0$$

which may be rewritten as

$$(\zeta_4 - \zeta_3)y^2 + (\zeta_7 - c'\zeta_2)y + (\zeta_8 - c'\zeta_3) = -(\zeta_5 y + (\zeta_6 - c'\zeta_1))\sqrt{-c' - y^2},$$

and further condensed notationally to

$$v_1 y^2 + v_2 y + v_3 = (v_4 y + v_5)\sqrt{-c' - y^2}. \quad (2-27)$$

Both sides of Eq. (2-27) are squared and rearranged to yield the quartic

$$(v_1^2 + v_4^2)y^4 + 2(v_1 v_2 + v_4 v_5)y^3 + (2v_1 v_2 + v_2^2 + c'v_4^2 + v_5^2)y^2 \\ + 2(v_2 v_3 + c'v_4 v_5)y + (v_3^2 + c'v_5^2) = 0. \quad (2-28)$$

Equation (2-28) may now be rooted for the solution values of y_i , $i = 1, 2, 3, 4$, and corresponding values of x_i found from

$$x_i = (v_1 y_i^2 + v_2 y_i + v_3)/(v_4 y_i + v_5) \quad (2-29)$$

The inverse of Eq. (2-25) may then be used to find $S(u,v)$; i.e.,

$$u_i = x_i - \frac{1}{2}a \\ v_i = y_i - \frac{1}{2}b. \quad (2-30)$$

Such solutions continue to be conceptually illustrated by Fig. II-6.

Five Multiply Separated Positions ($\ell = 1, 2, 3, 4$)

Consider the positions $\ell = 1, 2, 4$. This problem may be worked in precisely the same manner as that for $\ell = 1, 2, 3$, again resulting in a circular E-cubic. The points u, v which satisfy both problems are at the intersection of the two E-cubics. From the theory of algebraic curves, it is

known that a maximum of nine intersections exist. Two of these lie at infinity as the imaginary circle points I,J and are not useful. Indeed, a Bezoutian expansion for these two special E-cubics may be shown to yield a seventh order polynomial. Further, the kinematic problem has been shown to have three trivial intersections at the geometric poles P_{01} , P_{02} , P_{12}^0 , thus leaving a maximum of four such intersections which satisfy the stated problem. These four intersections are known as the Burmester points. Presuming that analogous trivial solutions occur in the spring problem (which is yet to be shown), and recognizing that the E-cubics have the same behavior at infinity as the kinematic cubics, there again must be four real solutions to satisfy the spring problem for $\ell = 1,2,3,4$. The fact that the spring problem involves deformable members allows the concurrent use of multiple solutions. This topic will be dealt with in detail in Chapters IV and V.

In order to formulate analytically the solution for the Spring Burmester points, it is recognized that Eq. (2-8) must be satisfied. This consists of two quadratics in u,v. A Bezoutian elimination may be carried out, resulting in a quartic in one variable which yields the desired solutions. To detail this process, Eq. (2-8) is again written as

$$Z_3 = Z_1 Z_5 + Z_2 Z_6$$

$$Z_4 = Z_1 Z_6 - Z_2 Z_5.$$

It is, however, recognized that Eq. (2-5) may be written* in the form

$$\sum_{m=3}^6 A_{m\ell} Z_m = -\sum_{m=0,1,2,7} A_{m\ell} Z_m, \quad \ell = 1,2,3,4$$

*The basis for the following notation was established by Tesar [27] and utilized recently by Myklebust and Tesar [24].

or

$$\begin{bmatrix} A_{31} & A_{41} & A_{51} & A_{61} \\ A_{32} & A_{42} & A_{52} & A_{62} \\ A_{33} & A_{43} & A_{53} & A_{63} \\ A_{34} & A_{44} & A_{54} & A_{64} \end{bmatrix} \begin{Bmatrix} Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{Bmatrix} = - \begin{bmatrix} A_{01} & A_{11} & A_{21} & A_{71} \\ A_{02} & A_{12} & A_{22} & A_{72} \\ A_{03} & A_{13} & A_{23} & A_{73} \\ A_{04} & A_{14} & A_{24} & A_{74} \end{bmatrix} \begin{Bmatrix} 1 \\ Z_1 \\ Z_2 \\ Z_7 \end{Bmatrix}.$$

More compactly, the matrix statement symbolically becomes

$$[\Delta] Z' = - [z] Z''$$

and hence the solution in terms of Z_1, Z_2, Z_7 is

$$Z_m = A_m + Z_1 B_m + Z_2 C_m + Z_7 D_m, \quad m = 3, 4, 5, 6 \quad (2-31)$$

where

$$A_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^4 A_{0\ell} a_{m\ell}$$

$$B_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^4 A_{1\ell} a_{m\ell}$$

$$C_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^4 A_{2\ell} a_{m\ell}$$

$$D_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^4 A_{7\ell} a_{m\ell}.$$

The $a_{m\ell}$'s are the determinants of the cofactors of the elements $A_{m\ell}$.

Substitution of Eq. (2-31) into Eq. (2-8) yields the two quadratics described earlier

$$av^2 + buv + cv + du^2 + eu + f = 0$$

$$a'v^2 + b'uv + c'v + d'u^2 + e'u + f' = 0$$

where

$$a = C_4, b = C_3 + B_4, C = A_4 + D_4 - C_1,$$

$$d = B_3, e = A_3 + D_3 - B_1, f = -A_1 - D_1$$

$$a' = C_3, b' = B_3 - C_4, c' = A_3 + D_3 + C_2,$$

$$d' = -B_4, e' = B_2 - A_4 - D_4, f' = A_2 + D_2$$

Utilizing the Bezoutian Eliminant, the resulting quartic in u is

$$F_4 u^4 + F_3 u^3 + F_2 u^2 + F_1 u + F_0 = 0$$

where

$$\begin{aligned} F_4 &= |ad'|^2 - |ab'| |bd'| \\ F_3 &= 2|ad'| |ae'| - |ab'| (|be'| + |cd'|) - |ac'| |bd'| \\ F_2 &= |ae'|^2 + 2|ad'| |af'| - |ab'| (|bf'| + |ce'|) \\ &\quad - |ac'| (|be'| + |cd'|) \\ F_1 &= 2|ae'| |af'| - |ac'| (|bf'| + |ce'|) = |ab'| |cf'| \\ F_0 &= |af'|^2 - |ac'| |cf'|. \end{aligned} \tag{2-32}$$

Here, Bezoutian notation is used for brevity as

$$|xy'| = xy' - x'y.$$

The solution for corresponding values of v_i for each of the values u_i , $i = 1, 2, 3, 4$ from the quartic is given by

$$v_i = -(|ad'| u_i^2 + |ae'| u_i + |af'|) / (|ab'| u_i + |ac'|). \tag{2-33}$$

Having thus obtained the values of coordinates u, v for the spring attachment point in E which satisfy the stated problem, the three-position transformation may again be invoked to solve for the corresponding coordinates

for the spring anchor $O_S(U_S, V_S)$ in Σ . Alternatively, it is possible to directly use (from Eq. (2-31))

$$U_S = -(A_3 + B_3 u + C_3 v + D_3 \delta)$$

$$V_S = -(A_4 + B_4 u + C_4 v + D_4 \delta).$$

Figure II-8 shows two such circular E-cubics for $\ell = 1, 2, 3$ and $\ell = 1, 2, 4$ and the calculated intersections.

Having found the Spring Burmester points, a return to the question of the three trivial points analogous to kinematic poles is now necessary.

For the five-position kinematic problem, the requirement is

$$\begin{vmatrix} \bar{D}_1 & E_1 & F_1 \\ \bar{D}_2 & E_2 & F_2 \\ \bar{D}_3 & E_3 & F_3 \end{vmatrix} = \begin{vmatrix} \bar{D}_1 & E_1 & F_1 \\ \bar{D}_2 & E_2 & F_2 \\ \bar{D}_4 & E_4 & F_4 \end{vmatrix} = 0.$$

Both of these circular cubics are trivially satisfied if

1. $\bar{D}_1 = E_1 = F_1 = 0$,
2. $\bar{D}_2 = E_2 = F_2 = 0$, or
3. $\bar{D}_1 = D_2, E_1 = E_2, F_1 = F_2$.

Conditions 1. and 2. are identically the requirements to locate the poles P_{01} and P_{02} respectively. The third condition locates the pole P_{12}^0 .

For the five-position spring problem, the requirement is

$$\begin{vmatrix} (\bar{D}_1 + \bar{G}_1) & E_1 & F_1 \\ (\bar{D}_2 + \bar{G}_2) & E_2 & F_2 \\ (\bar{D}_3 + \bar{G}_3) & E_3 & F_3 \end{vmatrix} = \begin{vmatrix} (\bar{D}_1 + \bar{G}_1) & E_1 & F_1 \\ (\bar{D}_2 + \bar{G}_2) & E_2 & F_2 \\ (\bar{D}_4 + \bar{G}_4) & E_4 & F_4 \end{vmatrix} = 0.$$

Again, the two E-cubics have the first two rows in common and it is natural to assume that a dependency between the two rows will yield trivial

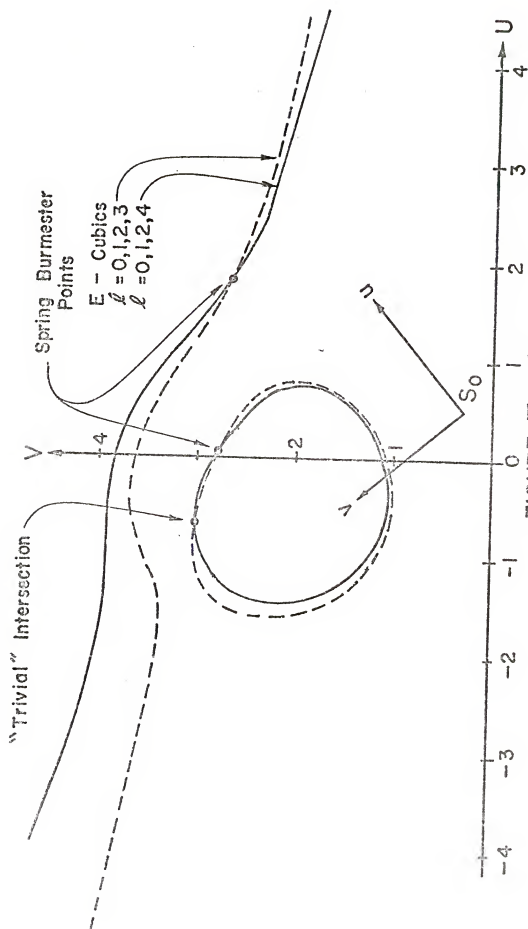


FIGURE II - 8

common points on both E-cubics. In keeping with this concept, assume that the first row is some unknown multiple λ of the second row; i.e.,

$$\begin{aligned}\bar{D}_1 + \bar{G}_1 &= \lambda(\bar{D}_2 + \bar{G}_2) \\ E_1 &= \lambda E_2 \\ F_1 &= \lambda F_2\end{aligned}\tag{2-34}$$

The problem is to now find the corresponding value(s) of λ which satisfy the above and to then determine the corresponding values of u, v in moving plane E which lie on the E-cubics as non-Burmester points. Equation (2-34) is thus expanded to

$$\begin{aligned}(A_{01} + \bar{G}_1) + A_{11}u + A_{21}v &= \lambda((A_{01} + \bar{G}_2) + A_{12}u + A_{22}v) \\ A_{51} + A_{31}u - A_{41}v &= \lambda(A_{52} + A_{32}u - A_{42}v) \\ A_{61} + A_{41}u + A_{31}v &= \lambda(A_{62} + A_{42}u + A_{32}v)\end{aligned}\tag{2-35}$$

which may be rewritten as

$$\begin{bmatrix} (A_{11} - \lambda A_{12}) & (A_{21} - \lambda A_{22}) & (A_{01} - \lambda A_{02} + \bar{G}_1 - \lambda \bar{G}_2) \\ (A_{31} - \lambda A_{32}) & (\lambda A_{42} - A_{41}) & (A_{51} - \lambda A_{52}) \\ (A_{41} - \lambda A_{42}) & (A_{31} - \lambda A_{32}) & (A_{61} - \lambda A_{62}) \end{bmatrix} \begin{Bmatrix} u \\ v \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

The determinant of the above matrix must vanish, resulting in an expression which is cubic in λ . The three roots are the values of λ being sought and are in agreement with the earlier conclusion that three trivial solutions must exist. To find the values of u, v which correspond to a given value of λ , any two of the expressions in Eq. (2-35) may be used after inserting the value of λ . The solution for u, v is then in terms of two linear equations in two unknowns.

Finally, it should be noted that the cubic in λ , when rooted, may yield imaginary solutions (there must be at least one real solution).

This in no way negates the validity of the development, just as the Burmester problem may yield 0, 2, or 4 imaginary solutions. It may also be demonstrated that for $\bar{G}_1 = \bar{G}_2 = 0$, the corresponding values of u, v are again the kinematic poles P_{01} , P_{02} , and P_{12}^0 .

CHAPTER III

ANALYTIC FORMULATION FOR INSERTION OF AN UNKNOWN SPRING CONSTANT INTO COPLANAR MOTION

The Controlling Equation

The material of the previous chapter was developed on the basis of known or specified motion, energy, and spring constant. The primary reason for assuming the spring constant to be known was to allow off-the-shelf component assemblies. It was shown that the limit of design based on closed-form procedures was five positions. Since this limit is the same for kinematic synthesis, it is even more desirable since any given motion could be concurrently synthesized; i.e., there was no need to assume that the mechanism pre-existed. It was further shown that when $K \rightarrow \infty$ or $\bar{G} \rightarrow 0$, then the problem reduces analytically to kinematic synthesis.

Similar techniques can, however, be developed to extend the preceding results to six-position design by allowing the spring constant to become an unknown. In general, the solutions will require that springs be custom-built. Of course, it is possible to work with fewer than six positions in which case it is still possible that the mechanism be concurrently synthesized. For the ultimate six-position problem, however, it is necessary that the mechanism pre-exist since general closed-form techniques are not available for six-position kinematic synthesis.

It will be helpful to develop these concepts on a parallel elementary basis. Thus, Eq. (2-10) is rewritten as

$$\bar{D}_\ell + Z_5 E_\ell + Z_6 F_\ell + Z_7 \bar{G}_\ell = 0 \quad (3-1)$$

where $\bar{G}_\ell$ now assumes the form

$$\bar{G}_\ell = A_7 \ell = \frac{d^k}{d\gamma^k} (\bar{E}_\ell - \bar{E}_0).$$

Equation (3-1) is now the controlling spring synthesis equation for $\ell = 1, 2, \dots$

Two Multiply Separated Positions ($\ell = 1$)

Presuming known or specified motion, the designer now has a number of choices as to how to utilize Eq. (3-1). Perhaps the most direct, as well as the most representative of the following formulation in this chapter, is to write

$$\delta \bar{G}_1 = U_S E_1 + V_S F_1 - \bar{D}_1. \quad (3-2)$$

Given the motion, the free choices are u, v, U_S, V_S, δ of which the designer must select four. From the form of Eq. (3-2), it is quite simple to pick u, v, U_S, V_S , and calculate δ . In effect, this means the designer may choose both sets of spring anchorage coordinates. For reasons previously detailed (see discussion in Chapter II for $\ell = 1$), the kinematic pole, P_{01} , is still excluded.

Three Multiply Separated Positions ($\ell = 1, 2$)

Equation (3-1) is now written twice to establish the three-position problem as

$$\begin{aligned} U_S E_1 + V_S F_1 - \delta \bar{G}_1 &= \bar{D}_1 \\ U_S E_2 + V_S F_2 - \delta \bar{G}_2 &= \bar{D}_2 \end{aligned} \quad (3-3)$$

Again, since the objective of this chapter is to deal with the spring constant as an unknown, the implication is that either U_s or V_s , as well as $S(u,v)$, must be chosen. Generally speaking, this case is not of great importance.

Four Multiply Separated Positions ($\ell = 1,2,3$)

Equation (3-1) is now written as

$$\begin{aligned} E_1 U_s + F_1 V_s - \bar{G}_1 \delta &= \bar{D}_1 \\ E_2 U_s + F_2 V_s - \bar{G}_2 \delta &= \bar{D}_2 \\ E_3 U_s + F_3 V_s - \bar{G}_3 \delta &= \bar{D}_3. \end{aligned} \tag{3-4}$$

The above does not normally represent a set of homogeneous equations since $\bar{D}_\ell = 0$, $\ell = 1,2,3$ requires

$$\begin{vmatrix} A_{01} & A_{11} & A_{21} \\ A_{02} & A_{12} & A_{22} \\ A_{03} & A_{13} & A_{23} \end{vmatrix} = 0$$

which is dependent solely upon the motion specifications. Hence Eq.

(3-4) merely represents a non-homogeneous set of three linear equations in three unknowns where the stipulation must be made that $|E_\ell \ F_\ell \ \bar{G}_\ell| \neq 0$. It may readily be demonstrated that $|E_\ell \ F_\ell \ \bar{G}_\ell| = 0$ is a circle locus in u,v and it is this locus which must be excluded from consideration of choices of $S(u,v)$ since Eq. (3-4) would then be singular.

The greatest importance of Eq. (3-4) is that it represents the four-position transformation of $u,v \rightarrow U_s, V_s, \delta$. It is completely analogous to the three-position transformation $u,v \rightarrow U_s, V_s$ for the case where the spring constant is known.

Five Multiply Separated Positions ($\ell = 1, 2, 3, 4$)

Equation (3-1) may now be written as

$$[E_\ell \ F_\ell \ -\bar{G}_\ell \ -\bar{D}_\ell] \begin{Bmatrix} u_s \\ v_s \\ \delta \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3-5)$$

requiring, for non-trivial solutions, that

$$|\bar{D}_\ell \ E_\ell \ F_\ell \ \bar{G}_\ell| = 0. \quad (3-6)$$

This may be expanded by the last column to yield

$$\begin{aligned} & \bar{G}_1 |\bar{D}_\ell \ E_\ell \ F_\ell|_{\ell=2,3,4} - \bar{G}_2 |\bar{D}_\ell \ E_\ell \ F_\ell|_{\ell=1,3,4} \\ & + \bar{G}_3 |\bar{D}_\ell \ E_\ell \ F_\ell|_{\ell=1,2,4} - \bar{G}_4 |\bar{D}_\ell \ E_\ell \ F_\ell|_{\ell=1,2,3} = 0. \end{aligned} \quad (3-7)$$

All the determinants in Eq. (3-7) are recognized as kinematic circular cubics, while the $\bar{G}_\ell$ are energy specifications and hence, numbers. Thus from the theory of algebraic curves, Eq. (3-7) is itself a circular cubic in u, v ; that is, one may write

$$\begin{aligned} & \bar{G}_1 \{ (u^2 + v^2) (H_{11}u + H_{21}v) + H_{31}u^2 + H_{41}v^2 + \dots \} \\ & - \bar{G}_2 \{ (u^2 + v^2) (H_{12}u + H_{22}v) + H_{32}u^2 + H_{42}v^2 + \dots \} \\ & + \bar{G}_3 \{ (u^2 + v^2) (H_{13}u + H_{23}v) + H_{33}u^2 + H_{43}v^2 + \dots \} \\ & - \bar{G}_4 \{ (u^2 + v^2) (H_{14}u + H_{24}v) + H_{34}u^2 + H_{44}v^2 + \dots \} = 0 \end{aligned}$$

or

$$(u^2 + v^2) (\underline{H}_1 u + \underline{H}_2 v) + \underline{H}_3 u^2 + \underline{H}_4 v^2 + \dots = 0, \quad (3-8)$$

where

$$H_i = \sum_{j=1}^4 (-1)^j H_{ij} \bar{G}_j.$$

Selection of any particular $S(u,v)$ which satisfies Eq. (3-8) means that $\bar{D}_\ell$, E_ℓ , F_ℓ , and $\bar{G}_\ell$ are known. For any three values of ℓ , this information may be substituted into the four-position transformation (Eq. (3-4)) to determine U_s , V_s , δ . As in Chapter II, the locus of all corresponding $O_s(U_s, V_s)$ is again a circular cubic, the Σ -cubic. It should be pointed out that concurrent kinematic synthesis is still feasible. This follows from the assumption that motion specifications are given and five-position design is being considered. Further, the five-position kinematic Burmester points identically satisfy Eq. (3-7). They do so regardless of the values of $\bar{G}_\ell$. This means that for any mechanism, whether concurrently synthesized or pre-existing, the pin joint locations in themselves automatically satisfy any energy curve specifications. The initial reaction is to presume that the resulting implication is one of "adjacent link" design and that the return to the four-position transform (Eq. (3-4)) must resolve the kinematic fixed pivot and spring anchor. Such is not the case, however, as may be demonstrated by conceiving four position values, say $\ell = 1, 2, 3$, returned to Eq. (3-4) to solve for U_s , V_s , δ . This means that the solution cannot be dependent upon $\bar{G}_4$ or, stated another way, the solution obtained is not affected by the choice of $\bar{G}_4$. Hence, one cannot expect to satisfy an arbitrary choice of $\bar{G}_4$, showing that such a process is not viable. Of course, it should be pointed out that the elusive "adjacent link" solutions are not the only solutions for the five-position problem.

Six Multiply Separated Positions ($\ell = 1, 2, 3, 4, 5$)

For $\ell = 1, 2, 3, 5$, an E-cubic similar to Eq. (3-7) may be written as

$$\begin{aligned} & \bar{G}_1 |\bar{D}_\ell E_\ell F_\ell|_{\ell=2,3,5} - \bar{G}_2 |\bar{D}_\ell E_\ell F_\ell|_{\ell=1,3,5} \\ & + \bar{G}_3 |\bar{D}_\ell E_\ell F_\ell|_{\ell=1,2,5} - \bar{G}_5 |\bar{D}_\ell E_\ell F_\ell|_{\ell=1,2,3} = 0. \end{aligned} \quad (3-9)$$

As suggested for the E-cubics developed in Chapter II, the intersections of Eqs. (3-7,9) should be the solutions for the six-position problem. There are trivial intersections, however. These will be dealt with later in this section. The meaningful solutions again must come from the requirement that Eq. (2-8) be satisfied. Equation (2-5) is rewritten as

$$\sum_{m=3}^7 A_{m\ell} Z_m = - \sum_{m=0}^2 A_{m\ell} Z_m, \quad \ell = 1, 2, 3, 4, 5$$

or

$$\begin{bmatrix} A_{31} & A_{41} & A_{51} & A_{61} & A_{71} \\ A_{32} & A_{42} & A_{52} & A_{62} & A_{72} \\ & & \cdot & & \\ & & \cdot & & \\ A_{35} & A_{45} & A_{55} & A_{65} & A_{75} \end{bmatrix} \begin{Bmatrix} Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \end{Bmatrix} = - \begin{bmatrix} A_{01} & A_{11} & A_{21} \\ A_{02} & A_{12} & A_{22} \\ & \cdot & \\ & \cdot & \\ A_{05} & A_{15} & A_{25} \end{bmatrix} \begin{Bmatrix} 1 \\ Z_1 \\ Z_2 \end{Bmatrix}.$$

In matrix symbology, this can be given as

$$[\Delta]Z' = -[Z]Z''$$

which has the explicit solution for $Z_3, \dots, Z_7$ in terms of Z_1, Z_2 as

$$Z_m = A_m + Z_1 B_m + Z_2 C_m, \quad m = 3, 4, 5, 6, 7 \quad (3-10)$$

where

$$A_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^5 A_0 \ell a_{m\ell}$$

$$B_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^5 A_1 \ell a_{m\ell}$$

$$C_m = - \frac{1}{|\Delta|} \sum_{\ell=1}^5 A_2 \ell a_{m\ell}$$

and the $a_{m\ell}$'s are the determinants of the cofactors of the elements, $A_{m\ell}$. Substitution of Eq. (3-10) into Eq. (2-8) yields the two quadratics

$$av^2 + buv + cv + du^2 + eu + f = 0$$

$$a'v^2 + b'uv + c'v + d'u^2 + e'u + f' = 0$$

where

$$a = C_4, b = C_3 + B_4, c = A_4 - C_1,$$

$$d = B_3, e = A_3 - B_1, f = -A_1 \quad (3-11)$$

$$a' = C_3, b' = B_3 - C_4, c' = A_3 + C_2$$

$$d' = -B_4, e' = B_2 - A_4, f' = A_2.$$

Use of the Bezoutian Eliminant again yields the quartic

$$F_4 u^4 + F_3 u^3 + F_2 u^2 + F_1 u + F_0 = 0,$$

where the F_j continue to be defined as in Eq. (2-31), but utilizing $a, b, \dots, a', b', \dots$ as defined in Eq. (3-11). Again there must be a maximum of four real solutions which satisfy the stated problem, and the corresponding values of v continue to be given by Eq. (2-32). Having obtained

the values of u, v , the four-position spring transformation (Eq. (3-4)) may be used to solve for the corresponding U_s, V_s, δ . Alternatively, direct utilization of Eq. (3-10) gives

$$U = -(A_3 + B_3u + C_3v)$$

$$V = -(A_4 + B_4u + C_4v)$$

$$\delta = (A_5 + B_5u + C_5v).$$

Because a six-position problem is under consideration, it is necessary that the mechanism pre-exist. This in turn means that the kinematic Burmester points for $\ell = 1, 2, 3, 4$ and $\ell = 1, 2, 3, 5$ are identical. Just as in the preceding section, these Burmester points must identically satisfy both Eq. (3-7) and Eq. (3-9) and are, for the same reasons developed earlier, not useful. Hence, of the four solutions corresponding to roots of the Burmester spring quartic, two of the solutions must be the kinematic Burmester points and cannot be utilized. This in turn means that a maximum of two viable solutions remain. Further, while a proof has not been developed, practical attempts to find two such solutions have always produced the result in which one of the solutions corresponds to a negative spring constant. The conclusion is that the six-position problem can never have more than one real solution, meaning that "multiple spring" solutions are not possible.

At this point, it is again necessary to account for the remaining three intersections of the two cubic loci. The loci may be written as

$$\begin{vmatrix} \bar{G}_1 & \bar{D}_1 & E_1 & F_1 \\ \bar{G}_2 & \bar{D}_2 & E_2 & F_2 \\ \bar{G}_3 & \bar{D}_3 & E_3 & F_3 \\ \bar{G}_4 & \bar{D}_4 & E_4 & F_4 \end{vmatrix} = \begin{vmatrix} \bar{G}_1 & \bar{D}_1 & E_1 & F_1 \\ \bar{G}_2 & \bar{D}_2 & E_2 & F_2 \\ \bar{G}_3 & \bar{D}_3 & E_3 & F_3 \\ \bar{G}_5 & \bar{D}_5 & E_5 & F_5 \end{vmatrix} = 0. \quad (3-12)$$

The trivial intersections should result from interrelations within the first three rows. For specified $\bar{G}_\ell$, $\ell = 1, 2, 3$, the singularity will depend on coefficients λ_i , $i = 2, 3$ which are found by

$$\bar{G}_1 = \lambda_2 \bar{G}_2 = \lambda_3 \bar{G}_3. \quad (3-13)$$

In Eq. (3-12), multiplication of the second row by λ_2 and subtraction from the first row yields the elements

$$0, (\bar{D}_1 - \lambda_2 \bar{D}_2), (E_1 - \lambda_2 E_2), (F_1 - \lambda_2 F_2). \quad (3-14)$$

Similarly for the first and third row, the elements obtained are

$$0, (\bar{D}_1 - \lambda_3 \bar{D}_3), (E_1 - \lambda_3 E_3), (F_1 - \lambda_3 F_3). \quad (3-15)$$

Expression (3-14) is now required to be a multiple of expression (3-15); i.e.,

$$\begin{aligned} (\bar{D}_1 - \lambda_2 \bar{D}_2) &= \lambda (\bar{D}_1 - \lambda_3 \bar{D}_3) \\ (E_1 - \lambda_2 E_2) &= \lambda (E_1 - \lambda_3 E_3) \\ (F_1 - \lambda_2 F_2) &= \lambda (F_1 - \lambda_3 F_3) \end{aligned} \quad (3-16)$$

where it should be emphasized that λ_2, λ_3 are fixed by Eq. (3-13). Equation (3-16) is expanded to

$$\begin{bmatrix} ((1 - \lambda)A_{01} - \lambda_2 A_{02} + \lambda \lambda_3 A_{03}) & ((1 - \lambda)A_{11} - \lambda_2 A_{12} + \lambda \lambda_3 A_{13}) \\ ((1 - \lambda)A_{51} - \lambda_2 A_{52} + \lambda \lambda_3 A_{53}) & ((1 - \lambda)A_{31} - \lambda_2 A_{32} + \lambda \lambda_3 A_{33}) \\ ((1 - \lambda)A_{61} - \lambda_2 A_{62} + \lambda \lambda_3 A_{63}) & ((1 - \lambda)A_{41} - \lambda_2 A_{42} + \lambda \lambda_3 A_{43}) \end{bmatrix} \begin{bmatrix} (1 - \lambda)A_{21} - \lambda_2 A_{22} + \lambda \lambda_3 A_{23} \\ (\lambda - 1)A_{41} + \lambda_2 A_{42} - \lambda \lambda_3 A_{43} \\ (1 - \lambda)A_{31} - \lambda_2 A_{32} + \lambda \lambda_3 A_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ u \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

requiring that the determinant of the above matrix vanish which, in turn, results in a cubic in λ . For each real value of λ which satisfies the cubic, the corresponding values of u, v may be found using any two of the expressions from Eq. (3-16). These values of u, v must correspond to intersections of the two five-position circular E-cubics, but cannot be viable solutions for the six-position spring synthesis problem.

CHAPTER IV
ADDITIONAL FORMULATIONS

Specification of Reference Energy

From Chapter II, Eq. (2-4) is combined with the energy specification to yield for the reference position $\ell = 0$,

$$\begin{aligned} & \frac{1}{2}(\alpha_0^2 + \beta_0^2) + u(\alpha_0 \cos \gamma_0 + \beta_0 \sin \gamma_0) + v(-\alpha_0 \sin \gamma_0 + \beta_0 \cos \gamma_0) \\ & - [vV_S + uU_S](\cos \gamma_0) + [vU_S - uV_S](\sin \gamma_0) - U_S(\alpha_0) - V_S(\beta_0) \\ & + \frac{1}{2}(U_S^2 + V_S^2 + u^2 + v^2) + \delta E_0 = 0. \end{aligned}$$

This may be written as

$$\frac{1}{2}(U_S^2 + V_S^2) + U_S(\epsilon_0) + V_S(\zeta_0) + v_0 = 0 \quad (4-1)$$

where

$$\begin{aligned} \epsilon_0 &= -u \cos \gamma_0 + v \sin \gamma_0 - \alpha_0 \\ \zeta_0 &= -u \sin \gamma_0 - v \cos \gamma_0 - \beta_0 \\ v_0 &= u(\alpha_0 \cos \gamma_0 + \beta_0 \sin \gamma_0) + v(-\alpha_0 \sin \gamma_0 + \beta_0 \cos \gamma_0) \\ &+ \frac{1}{2}(u^2 + v^2) + \frac{1}{2}(\alpha_0^2 + \beta_0^2) + \delta E_0 \end{aligned} \quad (4-2)$$

The condition represented by Eq. (4-2) must be satisfied in conjunction with conditions resulting from the remaining positions.

Two Multiply Separated Positions ($\ell = 0, 1$)

Equation (2-11) is again written for reference

$$E_1 U_S + F_1 V_S = \bar{D}_1 + \bar{G}_1 \quad (2-11)$$

Presuming that the motion and energy are given, the choice of u, v may be made and Eq. (2-11) is solved for U_S in terms of V_S . This result is returned to Eq. (4-2) to yield a quadratic in V_S . This procedure is illustrated for the two finitely separated positions, $\ell = 0, 1$, in Fig. IV-1. Obviously, there is a maximum of two solutions for the stated problem.

Three Multiply Separated Positions ($\ell = 0, 1, 2$)

Again from Chapter II, positions $\ell = 1, 2$ may be represented by

$$\begin{aligned} E_1 U_S + F_1 V_S &= \bar{D}_1 + \bar{G}_1 \\ E_2 U_S + F_2 V_S &= \bar{D}_2 + \bar{G}_2. \end{aligned} \quad (2-13)$$

In this case, $S(u, v)$ is not chosen, but rather, Eq. (2-13) is solved as

$$\begin{aligned} U_S &= \frac{(\bar{D}_\ell + \bar{G}_\ell) F_\ell}{|E_\ell F_\ell|} \\ V_S &= \frac{E_\ell (\bar{D}_\ell + \bar{G}_\ell)}{|E_\ell F_\ell|} \quad \ell = 1, 2 \end{aligned}$$

Substitution of these results into Eq. (4-2) yields

$$\begin{aligned} &\frac{(\bar{D}_\ell + \bar{G}_\ell) F_\ell}{|E_\ell F_\ell|}^2 + \frac{E_\ell (\bar{D}_\ell + \bar{G}_\ell)}{|E_\ell F_\ell|}^2 \\ &+ 2 \frac{(\bar{D}_\ell + \bar{G}_\ell) F_\ell}{|E_\ell F_\ell|} \epsilon_0 + 2 \frac{E_\ell (\bar{D}_\ell + \bar{G}_\ell)}{|E_\ell F_\ell|} \zeta_0 + 2 v_0 = 0 \end{aligned}$$

or

$$\begin{aligned} &|(\bar{D}_\ell + \bar{G}_\ell) F_\ell|^2 + |E_\ell (\bar{D}_\ell + \bar{G}_\ell)|^2 + 2 |(\bar{D}_\ell + \bar{G}_\ell) F_\ell| |E_\ell F_\ell| \epsilon_0 \\ &+ 2 |E_\ell (\bar{D}_\ell + \bar{G}_\ell)| |E_\ell F_\ell| \zeta_0 + 2 |E_\ell F_\ell|^2 v_0 = 0. \end{aligned} \quad (4-3)$$

Equation (4-3) now represents the locus of $S(u, v)$ which may be used to satisfy the stated problem. It should be noted that $|E_\ell F_\ell|$ represents

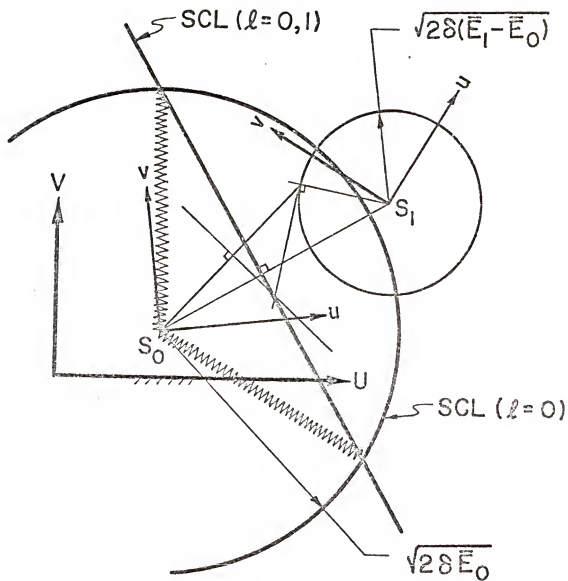


FIGURE IV - 1

a circle in u,v so that $|E_\ell F_\ell|^2$ is quartic. Hence the last term of Eq. (4-3) is of order six in u,v .

While the solution of such curves is not an insurmountable task, it is felt at present that such complex solution techniques are not warranted. There appear to be very few realistic problems in which absolute energy content is strictly required. Hence, the designer may indeed be in a far better position by utilizing techniques developed earlier which permit a range of energy reference levels. If he has some stipulation on, say, the maximum allowable reference energy, he may then search through the choices for the desired level. While the implication is that synthesis is to be partially discarded in favor of iteration, it should be noted that this is a directed choice among a limited set of good solutions; i.e., iteration is valid so long as there is a good probability of reaching the desired condition with a modicum of effort. To emphasize the preceding remarks, consider what has been accomplished in Chapter II -- the designer may specify positions $\ell = 1,2,3$ and arrive at the spring E-cubic, a curve which may readily be plotted or tabulated. A search may now be made along the E-cubic to try to find the $S(u,v)$ point(s) which satisfies the stipulated energy reference level. (If no such points exist, the designer must re-specify the stated problem.) Note that the current discussion really involves a four-position problem.

If, on the other hand, the designer must specify the reference energy level exactly and he wants all the solutions which satisfy the specifications, he may work with a sixth order locus of $S(u,v)$ and still only be guaranteed a three-position solution. This may, of course, be extended to four positions by intersecting the above-mentioned sixth order curve with the E-cubic (again, a process not guaranteed to produce solutions). In general, this process is not recommended.

Direct Specification of Torque

Quite often, the actual requirements to be satisfied are those of specific torque values at the known angular input. The processes of Chapters II and III apparently apply only to energy specifications, which means that the overall perspective of the problem has been clouded in the process of integration of the torque curve. The preceding material does, of course, allow torque specifications, but only in conjunction with the aforementioned energy terms. Hence, the best specific torque balancing which may be accomplished is to design a six-position energy curve of the type PP-PP-PP. This in turn appears to mean three "precision points" in the torque approximation. However, consideration of Fig. IV-2 shows by the "mean value theorem" that intermediate torque values must be satisfied. In this form, it is unfortunately not known, or specifiable, as to precisely where these pseudo-precision points will occur; i.e., it is only known that somewhere within the intervals (a) and (b) the slopes must be the same for the original and approximating curves. Since the specification of energy levels may not be of importance, can the design be accomplished by working directly with torque?

In order to investigate this question, the energy storage is again written with the requirement that energy removed from the spring is given up to the remainder of the system and hence

$$L_k^2 + 2\delta E_k = 0.$$

For the purposes of the present treatment, it is assumed that the required torque is for the angular parameter, γ , and the above is differentiated with respect to γ to yield

$$L_k I_k \dot{\gamma} + \delta T_k = 0.$$

— DESIRED CURVE
- - - APPROXIMATING CURVE
● PRECISION POINTS

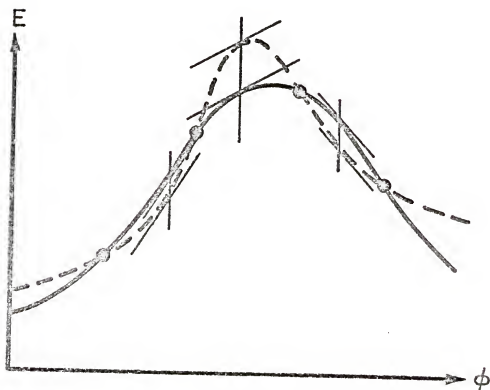


FIGURE IV - 2

Recall, however, that

$$L_{\ell}^2 = (U_{\ell} - U_S)^2 + (V_{\ell} - V_S)^2 \quad (2-1)$$

so that

$$L_{\ell} L'_{\ell} = (U_{\ell} - U_S) U'_{\ell} + (V_{\ell} - V_S) V'_{\ell} \quad (4-5)$$

and that

$$\begin{aligned} U_{\ell} &= u \cos \gamma_{\ell} - v \sin \gamma_{\ell} + \alpha_{\ell} \\ V_{\ell} &= u \sin \gamma_{\ell} + v \cos \gamma_{\ell} + \beta_{\ell} \end{aligned} \quad (2-2)$$

and hence

$$\begin{aligned} U'_{\ell} &= -u \sin \gamma_{\ell} - v \cos \gamma_{\ell} + \alpha'_{\ell} \\ V'_{\ell} &= u \cos \gamma_{\ell} - v \sin \gamma_{\ell} + \beta'_{\ell} . \end{aligned} \quad (4-6)$$

Equations (2-2, 4-6) are substituted into Eq. (4-5) and that result into Eq. (4-4) to yield

$$\begin{aligned} &(\alpha_{\ell} \alpha'_{\ell} + \beta_{\ell} \beta'_{\ell}) + u(-\alpha_{\ell} \sin \gamma_{\ell} + \beta_{\ell} \cos \gamma_{\ell}) \\ &+ v(-\alpha_{\ell} \cos \gamma_{\ell} - \beta_{\ell} \sin \gamma_{\ell}) + [uU_S + vV_S](\sin \gamma_{\ell}) \\ &+ [vV_S - uV_S](\cos \gamma_{\ell}) - U_S(\alpha'_{\ell}) - V_S(\beta'_{\ell}) + \delta T_{\ell} = 0. \end{aligned} \quad (4-7)$$

Letting the unknowns continue to be defined by Eq. (2-7), it is apparent that the required coefficients are simply the first derivative ($k=1$) of the $A_{m\ell}$ coefficients defined by Eq. (2-6). Indeed, it is valid for the reader to question the necessity of separately writing the equations for a case apparently already implied by the previous developments of Chapters II and III. The answer lies in that the main thrust is to remove the normally conceived displacement dependency of the analytic development.

A further clarification is perhaps in order. When a typical kinematic problem is being synthesized, the $A_{m\ell}$'s must be created on a displacement (or positional) basis; i.e., it is normally not possible to specify the geometric velocity content of the mechanism without also being able to locate the mechanism. The same is true, of course, for the higher geometric derivatives. For the spring problem, however, the mechanism may always be presumed to pre-exist regardless of whether or not such is actually the case. This in turn means that the coplanar motion is fully established and that any functional attachment may be made to the known motion set.

It should be noted that the derivation of Eq. (4-7) did not involve the subtraction of a reference position. This implies

1. reference information is not lost as was the case when working with energy terms, and
2. the same number of equations may be written as before (i.e., $\ell = 1, 2, \dots$). Since a reference position is no longer included, this in turn implies the loss of one position. Thus, the "best" case which may be treated directly for torque specifications is for an unknown spring constant and $\ell = 1, 2, 3, 4, 5$.

An associated benefit of this process is that no essentially new computing algorithms are necessary. The only stipulation is that the computer be informed that such a case is under investigation. This point will be further detailed in a later chapter.

Simultaneous Solutions

For most cases of spring synthesis, multiple solutions are the expected result. Each solution must satisfy the specified precision conditions but the behavior between the precision points is relatively

unpredictable. Momentarily considering finitely separated positions, the approximating function will normally exhibit zero order contact with the original function at the precision points. This in turn implies that, relative to the original function, the approximating function alternates between positive and negative error or "overshoot." For the various spring solutions, this "alternating" behavior may be opposite in sign. If so, the possibility immediately exists to utilize more than one solution since:

1. the use of multiple springs does not overconstrain the system, and
2. the approximating energy function is linearly dependent upon the spring constant.

Considering two such solutions, the spring constants may be multiplied by λ_1, λ_2 where $\lambda_1 + \lambda_2 = 1$ and $0 \leq \lambda_1, \lambda_2 \leq 1$. This guarantees that the inclusion of both springs continues to satisfy the precision conditions. Further, the relative error between precision points may be reduced. This concept is illustrated in Fig. IV-3 and an actual example is given in Chapter VI.

One may now conceive of an optimization scheme based upon the various solutions, S_j , requiring that

$$(\bar{E} - \sum_j \lambda_j S_j)_{\text{rms}} = \text{minimum}$$

where

$$\sum_j \lambda_j = 1, \quad |\lambda_j| \geq 0.$$

An alternate scheme may be

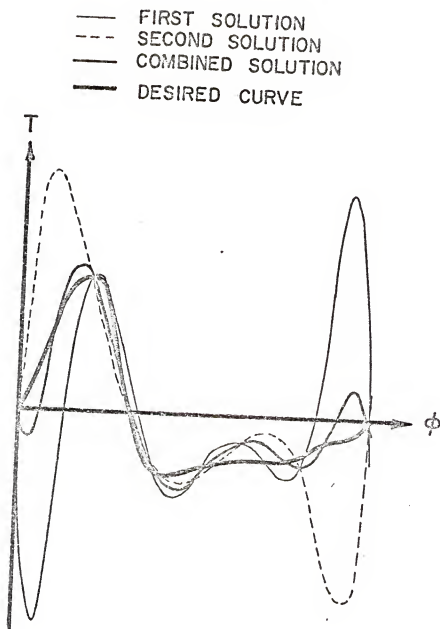


FIGURE IV - 3

$$(\bar{E} - \sum_j \lambda_j S_j)_{\max} - (\bar{E} - \sum_j \lambda_j S_j)_{\min} = \text{minimum.}$$

While these suggestions imply a return to optimization, it should be noted that the optimization is being conducted over a solution set already known to satisfy the required precision conditions. Further, the optimization process does not affect the precision conditions.

Another process to be considered is the simultaneous use of one solution from a given set of precision conditions and a second solution from a different set of precision conditions. While this technique may yield "good" results (dependent upon the stipulated quality criteria), there no longer is a guarantee that any of the precision conditions will be met.

Yet another use of simultaneous solutions may be devised by first considering one solution which satisfies the given precision conditions. A second spring may now be synthesized to cancel the errors produced by the first solution. Again, the original precision conditions will be lost. The sequential nature of this type of "filtered optimization" may be used in a backhand manner, however. For example, suppose that some energy or torque curve is required and that the precision points are invariant; i.e., no choice exists as to where the precision points are located. For the generation of the first solution, however, the designer purposely chooses precision points which do not coincide with the required precision points. The first solution is found, together with an associated error curve. The second solution is now generated by requiring that the error vanish at the original precision points.

As stated earlier, an example of simultaneous solutions will be presented in Chapter VI. It is, however, only for the purpose of

establishing the validity of the possible optimization concepts; i.e., optimization is not considered within the scope of this work and no extended treatment of the subject is to be developed.

CHAPTER V

DESIGN PROCEDURES

With the analytic formulation for spring synthesis well established, it now becomes necessary to elaborate upon the available techniques for the integration of springs into real mechanisms. Tables V-1 through V-4 show the various possible function classes and the respective requirements and results. For the actual techniques to be presented, discussion will involve torque rather than energy specifications, since it is felt that most designers will have a better intuitive understanding of torque concepts. Of course, it is understood that "torque specifications" may also mean derivatives of the torque curve, just as "motion" is meant to imply not only position, but also tangent, curvature, etc.

It should be recalled that the developments of Chapters II, III, and IV were based upon a notation closely associated with that of the kinematic $A_{m\ell}$'s. Thus, the torque function or, more accurately, the required precision points in torque, were required for various positions of the independent parameter, γ . This implies that the torque is acting on a link which exhibits general coplanar motion. For most real engineering applications, however, the torque is usually required at a known rotary link corresponding to an input shaft. The various methods of

1. properly transforming the torque to a rotary link, or
2. modifying the analytic formulation

will be the thrust of this chapter.

TABLE V -1: PARAMETER REQUIREMENTS AND
RESULTS FOR ENERGY FUNCTIONS
(KNOWN SPRING CONSTANT)

MSP CASE	REQUIRED SPRING PARAMETERS	RESULTS FROM SPRING SYNTHESIS
P-P-P, PP-P, PPP	u, v, K	U_s, V_s
P-P-P-P, PP-P-P, PP-PP	K	E - CUBIC IN u, v Σ - CUBIC IN U, V
P-P-P-P-P, PP-P-P-P-P PP-PP-P, PPP-P-P-P PPP-PP, PPPP-P PPPPP	K	SPRING BURMESTER POINTS IN u, v SPRING BURMESTER POINTS IN U, V (MAXIMUM OF 4 SOLUTIONS)

TABLE V-2: PARAMETER REQUIREMENTS AND RESULTS FOR ENERGY FUNCTIONS (UNKNOWN SPRING CONSTANT)

MSP CASE	REQUIRED SPRING PARAMETERS	RESULTS FROM SPRING SYNTHESIS
P-P-P-P, PP-P-P PP-PP, PPP-P, PPPP	u, v	K, U_s , V_s
P-P-P-P-P, PP-P-P-P PP-PP-P, PPP-P-P PPP-PP, PPPP-P PPPPP	—	E - CUBIC IN u, v Σ - CUBIC IN u, v CORRESPONDING VALUES OF K
P-P-P-P-P-P, PP-P-P-P-P PP-PP-P-P, PP-PP-PP PPP-P-P-P, PPP-PP-P PPP-PPPP, PPPP-P-P PPPP-PP, PPPPP-P PPPPPP	—	SPRING BURMESTER POINTS IN u, v SPRING BURMESTER POINTS IN u, v CORRESPONDING VALUE(S) OF K (USUALLY 1 SOLUTION)

TABLE V-3: PARAMETER REQUIREMENTS AND
RESULTS FOR DIRECT TORQUE FUNCTIONS
(KNOWN SPRING CONSTANT)

MSP CASE	REQUIRED SPRING PARAMETERS	RESULTS FROM SPRING SYNTHESIS
PP-PP, PPP	u, v, K	U_s, V_s
PP-PP-PP, PPP-PP, PPPP	K	E - CUBIC IN u, v Σ - CUBIC IN U, V
PP-PP-PP-PP, PPP-PP-PP PPP-PPP, PPPP-PP PPPPP		SPRING BURMESTER POINTS IN u, v SPRING BURMESTER POINTS IN U, V (MAXIMUM OF 4 SOLUTIONS)

TABLE V-4: PARAMETER REQUIREMENTS AND
RESULTS FOR DIRECT TORQUE FUNCTIONS
(UNKNOWN SPRING CONSTANT)

MSP CASE	REQUIRED SPRING PARAMETERS	RESULTS FROM SPRING SYNTHESIS
PP-PP-PP, PPP-PP, PPPP	u, v	U_s, V_s, K
PP-PP-PP-PP, PPP-PP-PP, PPP-PPP, PPPP-PP, PPPPP	—	E - CUBIC IN u, v Σ - CUBIC IN U, V CORRESPONDING VALUES OF K
PP-PP-PP-PP-PP, PPP-PP-PP-PP, PPP-PPP-PP, PPPP-PP-PP, PPPP-PPP PPPPP-PP, PPPPPP	—	SPRING BURMESTER POINTS IN u, v SPRING BURMESTER POINTS IN U, V CORRESPONDING VALUE(S) OF K (USUALLY 1 SOLUTION)

Besides dealing with energy or torque specifications, suggestions for other uses of springs are:

1. Simple energy storage with no rate (torque or power requirements). The spring is, in fact, one of the most efficient energy storage devices available to the practicing engineer (vis-a-vis chemical means such as the lead-acid battery, electrical means such as the capacitor, etc.). Springs have the further advantage of being able to store such energy over long periods of time with little deterioration.
2. Generation of prescribed forces for force balancing or moment balancing of mechanisms. Actually, the problem of shaking force balancing by redistribution of masses has been solved by Berkof and Lowen [28]. However, their technique generally tends to worsen the linkage torque requirements. A method for mass synthesis in linkages which is compatible with the Berkof and Lowen technique appears imminent. To the author's knowledge, however, there exists no general technique for moment balancing.
3. Generalization of torque concepts to spring balance generalized force-coordinate problems.

Some care is required in selecting the desired torque or energy curves. For example, if the input is to be cyclic, the energy in the spring at the start of one cycle must be the same as the energy in the spring at the start of the next cycle. This implies that the torque curve to be generated must have equal areas under the curve of opposite sign. Particularly if the desired torque is obtained by requiring that it negate an existing torque, one must be careful to specify the desired torque as the reflection of the pre-existing torque about its average as

shown in Fig. V-1. That such an average be non-zero is due to dissipative elements within the system.

Due to the non-linear nature of mechanisms, the need to consider an unusual driving torque requirement is the rule rather than the exception. Ability to torque balance at the existing "first assembly" level is thus desirable so that relative phasing between machine operations is predictable. In the case where it is desired to balance an aggregate machine, a linkage may be attached to the input shaft and the subsequent motion of the added linkage used to drive the spring element. Further, the added linkage may be arbitrarily selected by the designer, easing the design process, while allowing the maximum flexibility in arriving at the usable solution set.

For a pre-selected value of spring rate, the designer reduces the maximum number of design conditions by one. The set of unknowns is thus most likely to be well understood. Further, such springs are often obtainable on an "off-the-shelf" basis. On the other hand, allowing the spring constant to be unknown generally means that more accurate solutions can be obtained since more precision conditions may be specified.

It is suggested that the design methods given here are not a complete set and that new techniques may be easily devised for specific problems.

Path Cognates

Any four-bar mechanism may be initially chosen, so that α , β , γ for the specification of the coplanar motion are known from direct analysis as shown in Fig. V-2a. For various positions, the required torque values are supplied and the spring transformation applied, thus defining the spring parameters. The mechanism now resembles Fig. V-2b. In order to

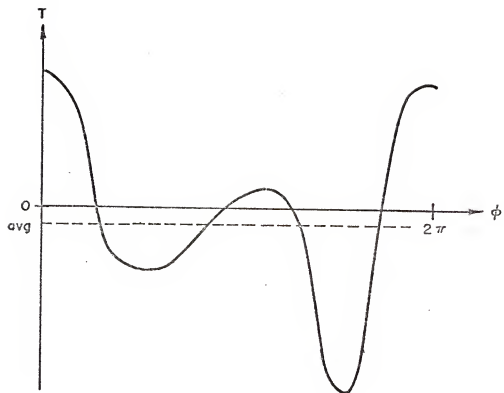


FIGURE V - 1

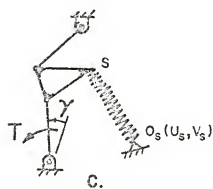
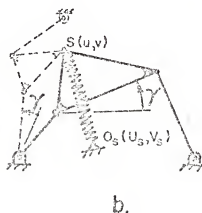
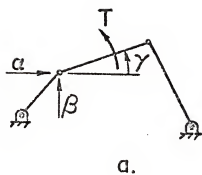
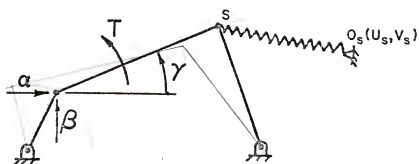


FIGURE V - 2

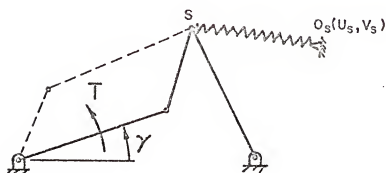
transform the torque to a rotary link, Roberts' Cognate principle is now applied for point S which transforms the torque function to a crank with the final result shown in Fig. V-2c. Because the point S is the only point in the moving plane which will exhibit the same point path properties as before the path cognate principle was utilized, this technique cannot be applied to a mechanism which has some additional point path requirement.

Angular Cognates

Again, any four-bar mechanism may be chosen initially. For 3 MSP, point S is chosen to occur at the moving pivot as shown in Fig. V-3a and the remaining spring parameters are found. Parallelogram construction as shown in Fig. V-3b is utilized to transfer the motion to the rotary link as input. Again, point path properties cannot be maintained. A special case of this technique results for 4 MSP adjacent link design. The required initial assumption is represented by Fig. V-4a and the simultaneous solutions of the E-cubic and the kinematic circle point curve yield point S. The spring attachment point, S, and the kinematic circle point, A, are coincident. This information is in turn used in the three-position transformation to calculate the remaining spring parameters, U_s, V_s for O_s . The fixed kinematic pivot for point A may also be found by setting all torque levels to zero to locate O_a . The result, as shown in Fig. V-4b, must also be subjected to angular cognates to transform the desired motion and torque to the rotary link. The final mechanism is shown in Fig. V-4c. Because the entire mechanism does not pre-exist, however, this is the one case which cannot guarantee that all positions are associated with a given closure or that the positions follow the desired sequencing.

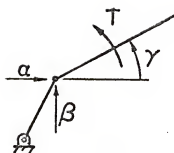


a.

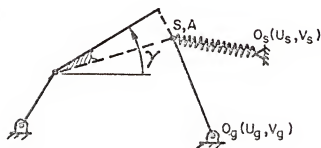


b.

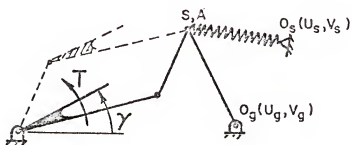
FIGURE V - 3



a.



b.



c.

FIGURE V - 4

Angular Inversion

When it is necessary that additional pre-existing functional requirements should not be disturbed by the inclusion of a spring, further effort appears necessary. Suppose such point path properties exist in the four-bar shown in Fig. V-5a, together with a torque requirement. Motion may then be inverted as shown in Fig. V-5b, spring parameters calculated, and a re-inversion carried out to place the torque at the required link as shown in Fig. V-5c. This technique is necessary whenever both ends of the spring are to be on moving links; i.e., an inversion may always be used which "fixes" one of the moving links in question. While this type of design may be quite productive in limiting internal pin joint forces, the inherent space restrictions are liable to cause difficulties in practical implementation.

Transform of Torque to New Link

For a pre-existing mechanism which already satisfies desired motion properties, perhaps from a prior kinematic synthesis effort, any link may be selected as a candidate for spring attachment. The required input torque may be transformed to the link in question by the modeling techniques of Benedict and Tesar [2]. Hence the torque function is known for given values of the angular parameter of the moving link. This process is depicted in Fig. V-6. While the designer is no longer specifically utilizing the original torque function, the result must effectively satisfy the torque function in the original location.

Change of Independent Parameter

With the exception of the unique design capabilities of the angular inversion technique, the difficulties encountered by the preceding methods

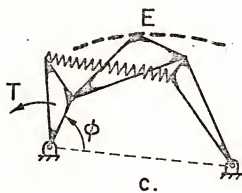
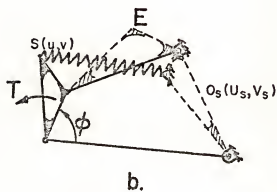
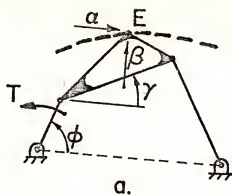


FIGURE V - 5

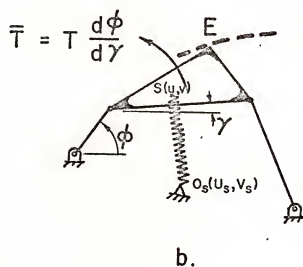
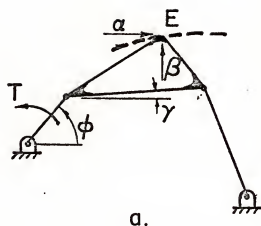


FIGURE V - 6

are brought about by the fact that the $A_{m\ell}$'s are written as functions of angular motion parameter, γ , of the moving plane, while the torque is required at a rotary link governed by another angular parameter. Myklebust and Tesar [24] recently developed the partial time state synthesis for mechanisms, requiring $\alpha = f(\tau)$, $\beta = f(\tau)$, $\gamma = f(\tau)$. The functional $A_{m\ell}$'s are then differentiated with respect to the dummy parameter, τ . In their case, the most advanced problem depended on $\tau = \text{TIME}$. Their analytical development is, however, immediately applicable to the spring problem by letting τ represent the angular rotation of the link where the desired torque occurs; i.e., ϕ as shown in Fig. V-7. In this case, no increased complexity in understanding occurs in order to utilize any link exhibiting coplanar motion, regardless of how "far removed" it is from the rotating input. When carrying out the differentiation with respect to τ ($=\phi$), the $A_{m\ell}$'s then become $T_{m\ell}$'s. For example, consider the coefficient $A_{m\ell}$ for the zeroth derivative, denoted temporarily by $A_{3\ell}^0$. The same form is used for $T_{3\ell}$; i.e., $T_{3\ell}^0 = A_{3\ell}^0 = \cos \gamma_\ell - \cos \gamma_0$. For the first derivative, $A_{3\ell}^1 = -\sin \gamma_\ell$ and $T_{3\ell}^1 = -\sin \gamma_\ell \frac{d\gamma_\ell}{d\phi}$. Similarly for the second derivative, $A_{3\ell}^2 = -\cos \gamma_\ell$ and $T_{3\ell}^2 = -\cos \gamma_\ell \frac{d\gamma_\ell}{d\phi}^2 - \sin \gamma_\ell \frac{d^2\gamma_\ell}{d\phi^2}$. It is the $T_{m\ell}$ technique which is used to generate all examples to be shown in Chapter VI and it is strongly felt that, with the exception of angular inversion, all previous techniques by comparison may be considered less fruitful.

Methods Requiring Kinematic Synthesis

The procedure described by Conley [1] for the utilization of torsion springs consists of pre-selecting the desired energy relationship and spring parameters -- in this case, torsional spring constant and pre-load.



FIGURE V - 7

The energy to be stored in the output link torsion spring is then fixed by specifying coplanar synthesis with motion along the assumed circular arc of the output link and angular motion of the coupler which can later be transferred from the coupler to the input via angular cognates. The procedure has proved quite satisfactory although somewhat limited in scope. Placement of the torsion spring at a joint other than at the frame can be accomplished by specifying the relative rotation between input and coupler to store the required energy and simply finding the proper kinematic constraint. This method is limited, however, to cases where complete rotation of the input is not required as for example in switches. Alternatively, one may wish to place the torsion spring between the coupler and follower links, which may more likely be accomplished by using the vector techniques of kinematic synthesis.

The problem of free length for linear springs is not considered within the scope of this work due to the extreme non-linearity of the controlling equations. A method similar to that of torsion springs where kinematic synthesis is the final goal is rather easily developed, however. This technique consists of specifying the total spring length on the basis of required energy criteria. A spring anchorage location is chosen ($(0,0)$ may be used with no loss of generality). Now a U-location for the moving end of the spring may be chosen and the V-location calculated from the stated length requirements. A coupler angle is also specified which is to eventually correspond to the input rotation. A linkage is now synthesized and transformed through the path cognate principle. The process is shown in Fig. V-8. While this technique does work, the ability to find satisfactory solutions is highly dependent upon designer experience. This is inherently due to the large number of free choices available to

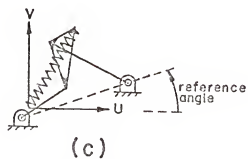
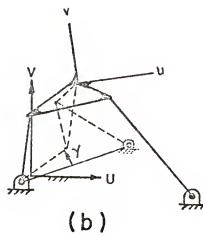
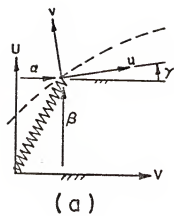


FIGURE V - 8

the designer at the outset. It must be emphasized that none of the techniques which require kinematic synthesis allow the direct utilization of torque. This point will be clarified in a later section of this chapter.

Intuitive Design

The case may arise where a required energy function is of a simplistic nature so that the formal synthesis process is unnecessary. For example, if an energy curve is required which is constant for some range of the independent input parameter, one may wish to consider the use of a mechanism known to produce a path as shown in Fig. V-9a. A spring is attached, yielding a compact, lightweight, and reliable device. For the requirement of two dwells, a mechanism may be selected which exhibits a path as shown in Fig. V-9b. Other elementary ideas are also conceivable, but will not be detailed here.

Direct Treatment of Torque

As developed in Chapter IV, the analytic formulation permits the designer to utilize direct torque specifications without consideration of the energy curve. This means that the $A_{m\ell}$'s (or $T_{m\ell}$'s) which are developed for the finitely separated positions are used only to gather the information about the state of the mechanism and that the energy specification is arbitrary. Subsequently, those $A_{m\ell}$'s which correspond to the finitely separated positions will be dropped. Referring to Table V-4, note that a treatable case is shown as PP-PP-PP. Dropping the finite positions for torque considerations leaves PP-P-P. For example, the calculated $A_{m\ell}$'s for the case PPP-PP-PP are

$$A_{m1} \ A_{m2} \ A_{m3} \ A_{m4} \ A_{m5} \ A_{m6} \ A_{m7}, \quad m = 0, 1, \dots, 7$$

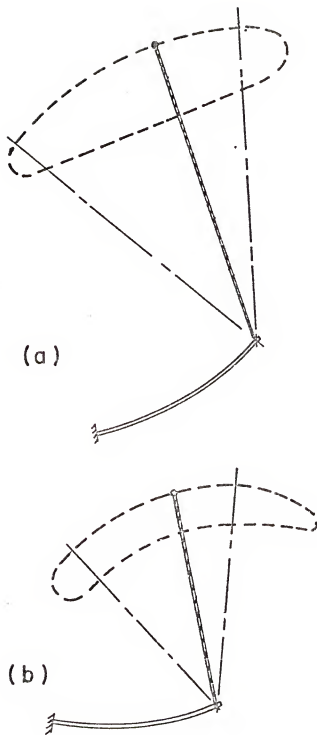


FIGURE V - 9

but the $A_{m\ell}$'s actually required in the computation are

$$A_{m2} A_{m3} A_{m5} A_{m7}, \quad m = 0, 1, \dots, 7.$$

The computer programs which have been developed interrogate the user by asking

"Are the data specified for torque only?"

If the answer is "yes", the coding automatically removes the $A_{m\ell}$'s for the finite positions. This subject of computer programming is again discussed in Appendix B.

Actual Spring Configurations

Because the analytical formulation is developed on the basis of a linear spring having a zero free length, it is necessary to examine some of the forms such springs may take. Springs may actually be wound in such a fashion that zero free length would be achieved if coil interference could be avoided. Since such interference does occur, there is a minimum usable spring length. The force-displacement curve for such a spring is shown in Fig. V-10. Quite often, this type of spring is limited in practical application because non-linearities quickly arise, also depicted in Fig. V-10.

In order to develop alternative physical forms, it is first necessary to recognize that zero free length means that no net force exists in the spring when u, v is coincident with U_S, V_S . It then becomes apparent that the inverted slider shown in Fig. V-11 is a feasible physical solution. Any real free length may then be chosen and the inverted slider mechanism built so that the spring is not displaced from its free length when u, v is coincident with U_S, V_S . An alternative physical form of the same technique is to place the oscillating slider on the moving plane.

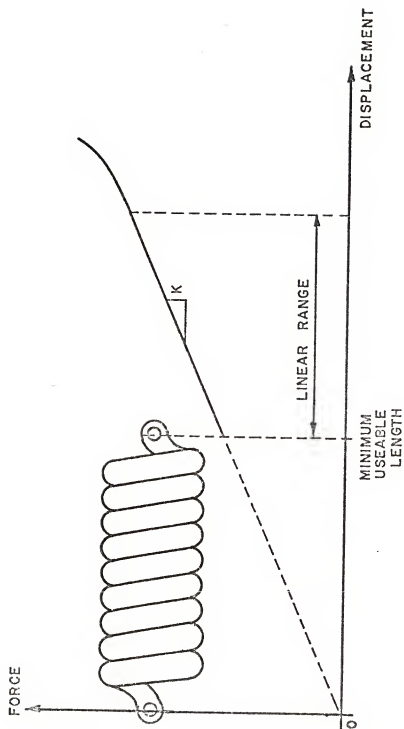


FIGURE V - 10

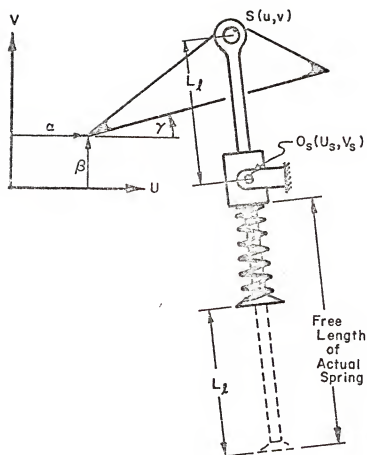


FIGURE V - II

The inability to specify or treat a non-zero free length does, however, result in disadvantages. Regardless of the physical forms shown previously, the implication is that all these springs are behaving as though they were extension springs. Considering that many common spring problems depend upon the use of compression springs, attempted replacement by extension springs may not yield reasonable solutions, particularly where space limitations are of paramount concern. Indeed, such difficulties are documented for the case of an automobile suspension, used as an example in Chapter VI.

CHAPTER VI
DESIGN EXAMPLES

In order to illustrate the preceding analytics, it is now useful to examine some typical applications. At the same time, some "feel" may be gained for the quality of the results that can be expected. In the following, all numerical quantities will have the associated units:

mass	lb _m .
inertia	lb _m .-ft ²
length	ft.
angles	degrees (°)
angular rates (ω)	radians/sec.
torque	ft.-lb _f .
spring constants	lb _f /ft.

Any exceptions will be noted. With one exception, all examples use the $T_{m\ell}$ * formulation described in Chapter V.

Cardan Drag Link

The Cardan Drag Link [29] is a mechanism useful for its particular speed variation qualities. For a constant speed input, the output angular velocity drops to zero during a portion of the cycle and smoothly accelerates to double the input speed, decelerates back to zero, etc., always remaining non-negative. Thus an assembly line can be decoupled from the remainder of the machine system, an arbitrary number of cycles allowed to elapse, and the assembly line brought back to operational speed. The Cardan Drag Link thus behaves as a clutch which, when engaged, guarantees that proper system phasing is maintained. The mechanism is in

*The $T_{m\ell}$'s are tabulated in Appendix C.

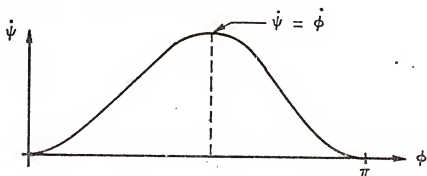
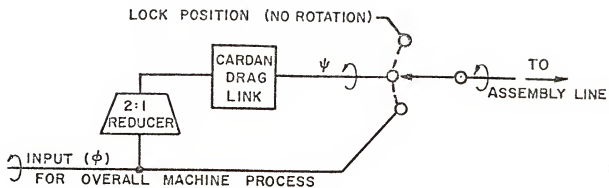
constant operation and a set of locking pins connect the output to the assembly line when the start-up process is initiated. With the assembly line at operating speed, the lock pins are transferred so that the assembly line is directly connected to the main machine input. A similar reverse process may be employed to decouple the assembly line from the remainder of the machine. A simplified schematic of the total operation is shown in Fig. VI-1a, while a kinematic representation of the Cardan Drag Link is shown in Fig. VI-1b. Two distinct contributions to torque variations immediately arise:

1. The Cardan mechanism contains links whose kinetic energy is not constant during the cycle (see links 1 and 2 in Fig. VI-1b).
2. A torque must be supplied to balance the inertial load of the assembly line acceleration.

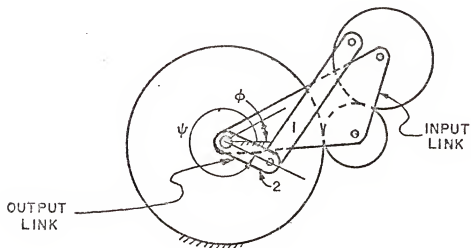
Since the Cardan mechanism is always in operation, a torque balance should be supplied which always counterbalances the first effect. To counterbalance the second effect, the balancing should take place whenever the assembly line is being accelerated.

Considering the latter problem first, an analysis of the Cardan output is performed to show that the torque exerted on the assembly line input shaft by the inertial load is represented by Fig. VI-2. Thus it is desired that a spring-augmented mechanism be synthesized which will produce the negative of the given torque, also shown in Fig. VI-2. Since it is required that this mechanism act only during assembly line acceleration, the mechanism may be arbitrarily chosen.

An energy approach was first considered, requiring that the desired torque of Fig. VI-2 be numerically integrated with respect to ϕ . Five finitely separated energy conditions were then supplied corresponding to



(a)



(b)

FIGURE VI - 1

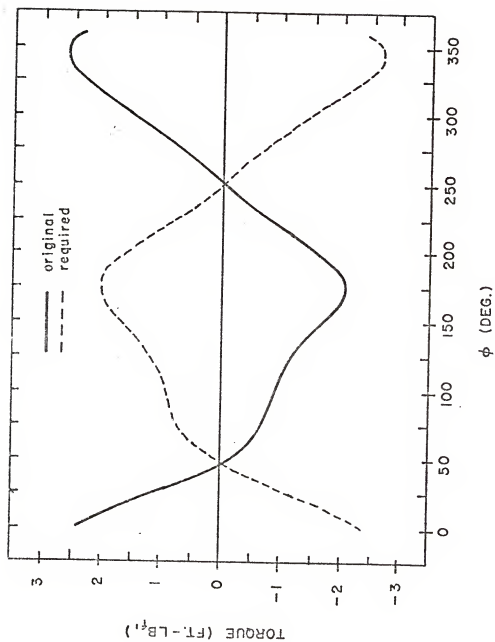


FIGURE VI - 2

$\phi = 30^\circ, 100^\circ, 170^\circ, 230^\circ, 270^\circ$. The spring constant was considered unknown and the E-cubic calculated. Two solutions from the E-cubic were selected as shown in Table VI-1.

TABLE VI-1: Five FSP (Energy) Solutions

Soln.	u	v	U_s	V_s	K
1	0.4393	-0.6734	0.49222	-1.4488	1.8833
2	4.3299	-1.4943	2.68583	3.0485	1.7004

These solutions are depicted in Fig. VI-3 and Fig. VI-4, respectively. An analysis was then conducted for the solutions, with results shown in Fig. VI-5 and Fig. VI-6. Note that the associated error curves are not zero at the specified values of ϕ since the problem was formulated as an energy balance while the results are shown for torque. A substantial improvement in torque variation has taken place for the first solution. Table VI-2 shows the maximum and minimum for the original torque as well as the error curves associated with the two solutions.

TABLE VI-2: Comparisons for Five FSP (Energy)

Soln.	Max.	Min.	Max.-Min.
(Orig.)	2.7	-1.6	4.3
1	1.6	-0.8	2.4
2	4.5	-5.5	10.0

Implementation of the second solution would actually be detrimental to the system. Because the errors of the two solutions are of opposite sense, however, it is immediately feasible to apply the "multiple solutions" concept discussed in Chapter IV. This has indeed been done, multiplying K_1 by 0.8 (giving $K_1 = 1.5066$) and K_2 by .2 (giving $K_2 = 0.3401$). The

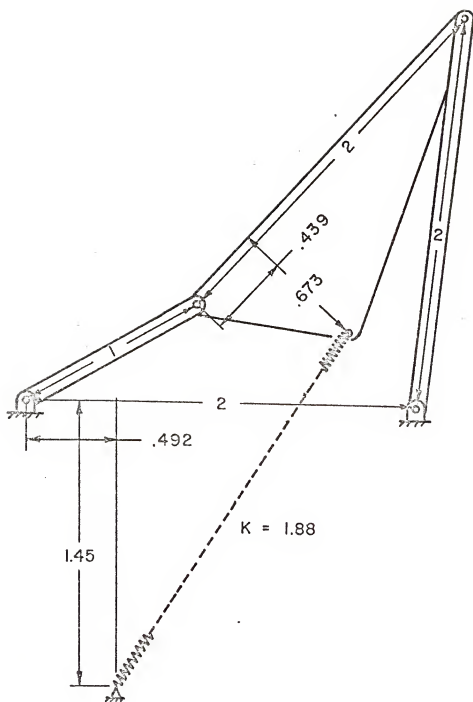


FIGURE VI - 3

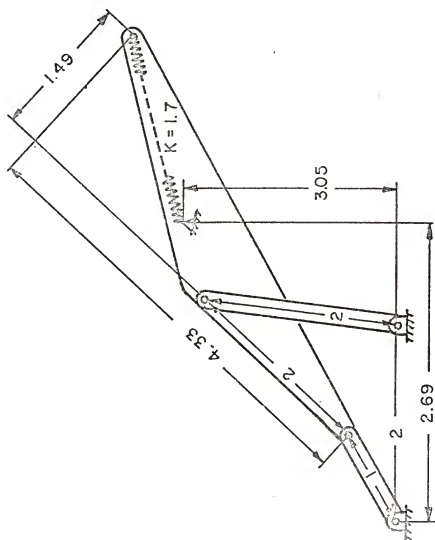


FIGURE VI - 4

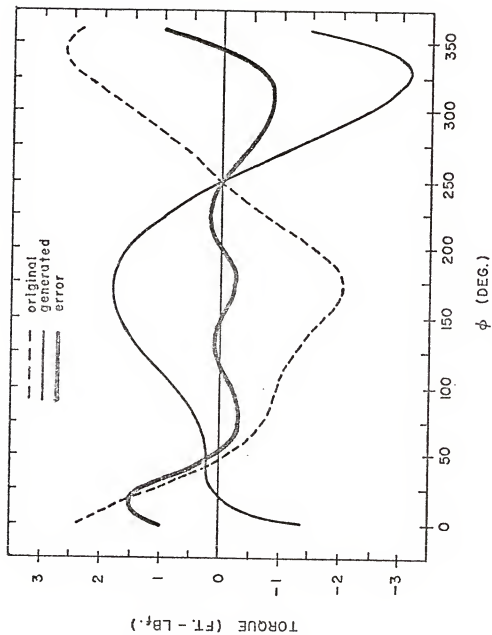


FIGURE VI - 5

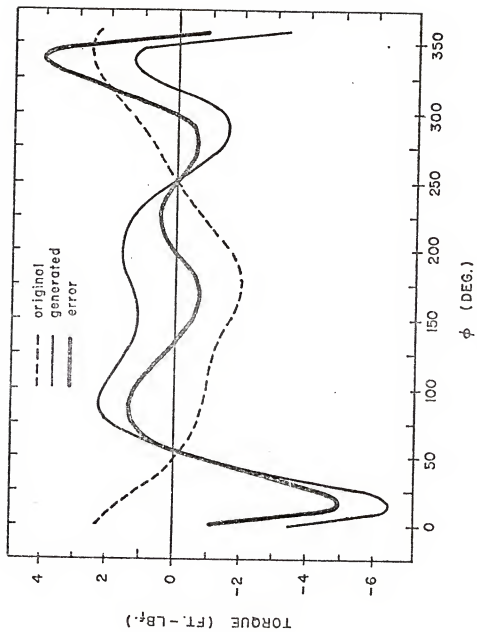


FIGURE VI - 6

combined solution is shown in Fig. VI-7, having an error curve with a maximum of 0.8 and a minimum of -0.7 for peak-to-peak variation of 1.5. This means that a 65% reduction in torque variation has been achieved.

It should be noted that in all cases, the worst error occurs during a portion of the cycle which corresponds to a large change of the independent parameter while no precision specifications were made. That is, from 30° to 270° , the data are specified at roughly 50° increments whereas the major error occurs during the rotation from 270° to 390° (i.e., $360^\circ + 30^\circ$). This is 120° or fully one-third of the cycle. The preceding remarks indicate that an immediate extension of the problem is the additional specification of a sixth position and the corresponding value of energy. The position $\phi = 320^\circ$ was chosen in addition to the former set and one Spring Burmester point found. The results are shown in Fig. VI-8 and Fig. VI-9 which, perhaps suprisingly, do not indicate an improvement over the previous result -- in fact, a peak-to-peak reduction of only 35% occurs.

Continuing with the same problem definition, an attempt is now made to balance via finitely separated positions using direct torque specifications. A number of 5 FSP* cases were attempted with no resulting Spring Burmester points. Solutions were obtained by switching to 4 FSP and reducing the length of the input crank for the spring mechanism by a factor of two. The precision points were specified to occur at $\phi = 30^\circ$, 130° , 210° , and 300° . The mechanism and results (again selected from a generated E-cubic) are shown in Fig. VI-10 and Fig. VI-11, respectively.

*It should again be emphasized that the actual motion is of the type PP-PP-PP-PP-PP for this case. Energy specifications are arbitrary.

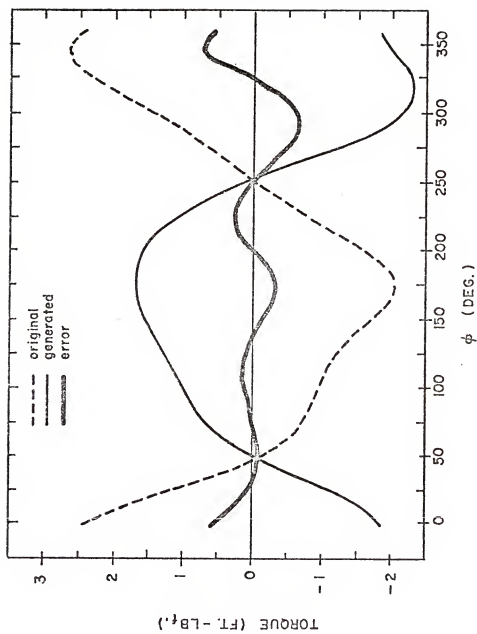


FIGURE VI - 7

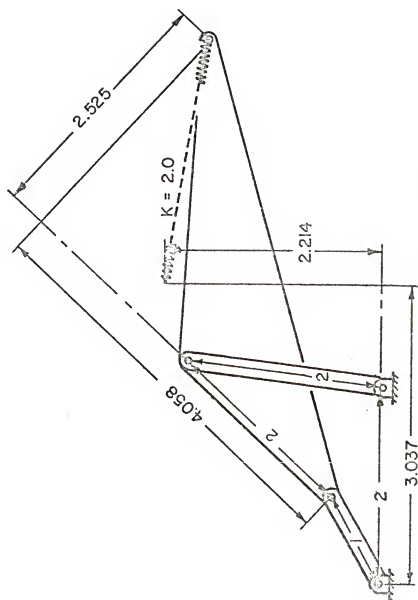


FIGURE VI - 8

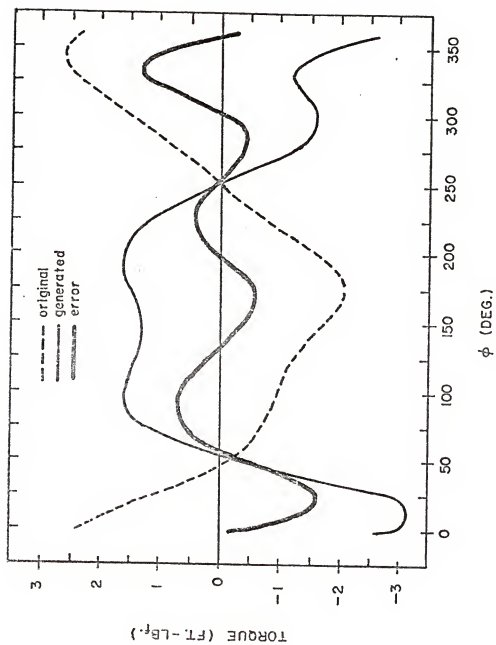


FIGURE VI - 9

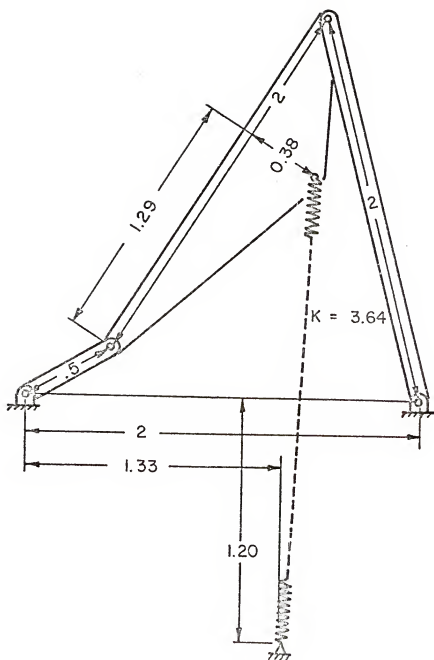


FIGURE VI - 10

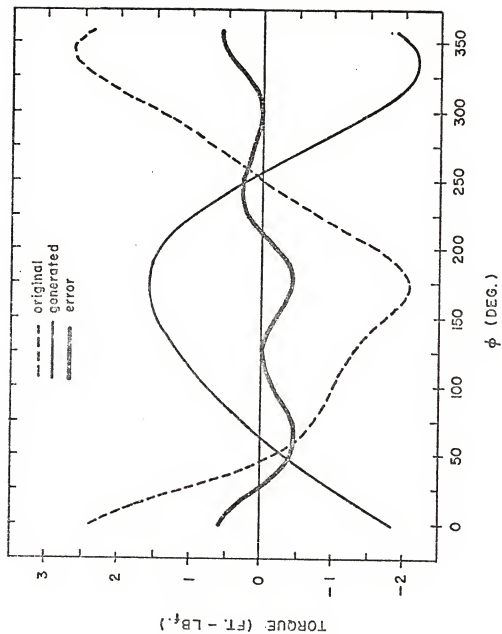


FIGURE VI - II

The peak-to-peak variation has been decreased to 1.25, corresponding to a 71% reduction.

For all previous cases, the problem definition producing the "given" torque curve included the assumptions that the effective assembly line moment of inertia was $I = 1$ and that the input speed was $\omega = 1$. Thus the energy was proportional to $I\omega^2 = 1$. For any other conditions, say $I\omega^2 = 100$, it is only necessary to multiply the various spring constant solutions by the factor of 100.

Returning now to the first problem listed for the Cardan Drag Link, the torque produced at the system input is shown in Fig. VI-12. This curve is the result of assuming that all gears have their mass centers at the respective geometric centers, that link 1 has a mass and inertia content of 2 and .135, respectively, and that link 2 has a mass and inertia content of 1 and .0675, respectively. Fig. VI-12 represents one-half of a Cardan cycle, but the two halves are identical. Hence, the torque balance will be attempted at the system input rather than the Cardan input since:

1. trying to balance twice as many cycles by associating the spring directly with the Cardan linkage is not practical, and
2. the Cardan Drag Link is such a compact physical device that the insertion of springs is likely to prove difficult.

Recalling the initial system schematic of Fig. VI-1a, the final system configuration, including the previous solution for output load balancing, may be represented by Fig. VI-13.

Returning to the torque curve of Fig. VI-12, a possible means of counterbalancing takes the form of specifying the negative of the two torque peaks and specifying the value at the horizontal inflection of

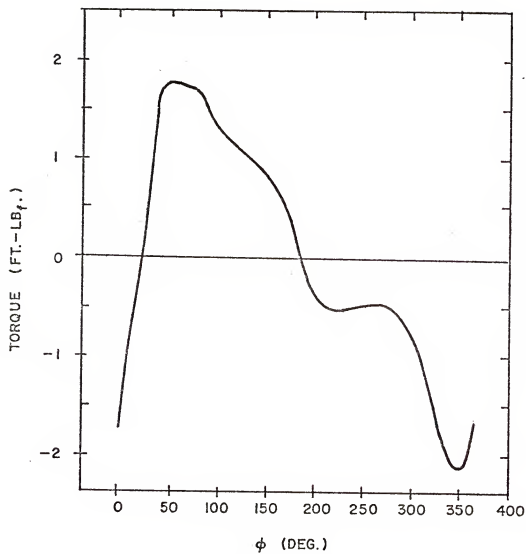


FIGURE VI - 12

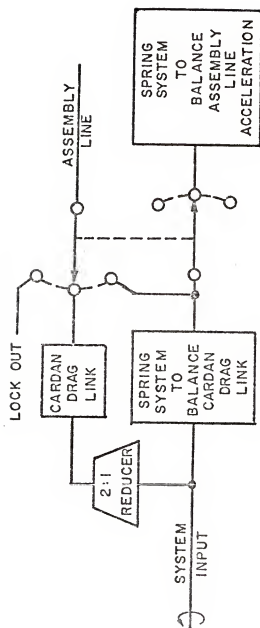


FIGURE VI - 13

the curve together with a slope T' , of zero. The resultant mechanism and torque curves are shown in Fig. VI-14 and Fig. VI-15, respectively. A 74% reduction of the peak-to-peak variation has been achieved. Note that the type of function approximation has been PP-P-P for torque, meaning PPP-PP-PP was specified for the motion considerations. A second solution may be found by requiring that the negative of the peaks and their associated slopes* be matched. The solution mechanism and attendant curves are shown in Fig. VI-16 and Fig. VI-17. In this case, results were not nearly as good as before, achieving a torque reduction of only 22%. Note that the function approximation has been PP-PP for torque, meaning PPP-PPP for the motion considerations.

It should be noted that the assumed torque curve is not invariant as was the case for the preceding examples; i.e., this example is the result of several assumptions of link masses and inertias as well as the operating speed. Hence, for actual implementation, the user must have specific knowledge of the parameter content in the particular Cardan Drag Link. Once known, however, the spring constant may again be proportioned to the square of the operating speed.

Automotive Suspension System

Suspension design on racing cars has long been of interest to designers in that field. Having somehow obtained the desired geometry, the next problem is the proper application of springs and shock absorbers. Unfortunately, the use of the term "proper" is a subjective one at best. Changing track conditions will often bring driver requests that spring-shock rates be adjusted. In fact, even when one driver is completely

*Because the data do not actually occur at true peak values, a graphical approximation of the non-zero slope was obtained.

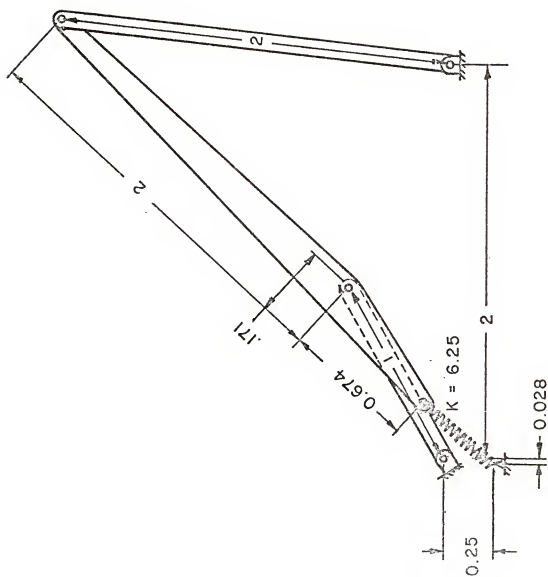


FIGURE VI - 14

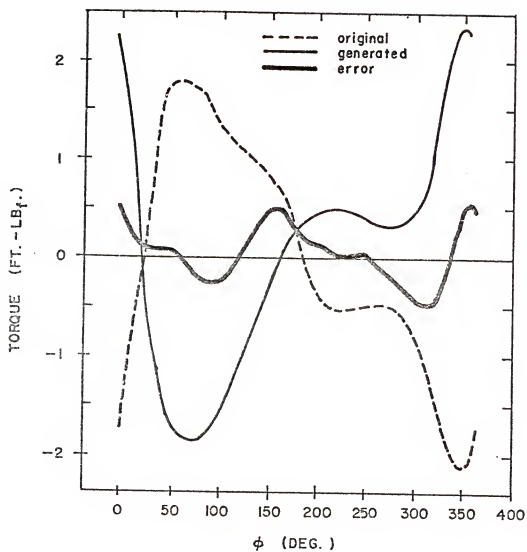


FIGURE VI - 15

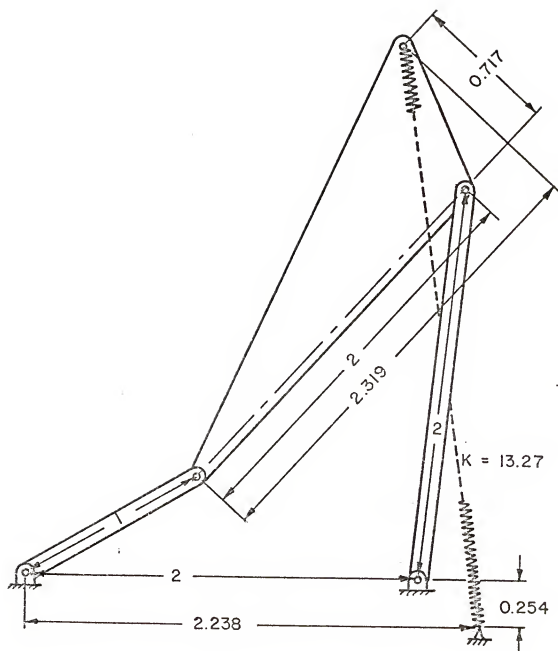


FIGURE VI - 16

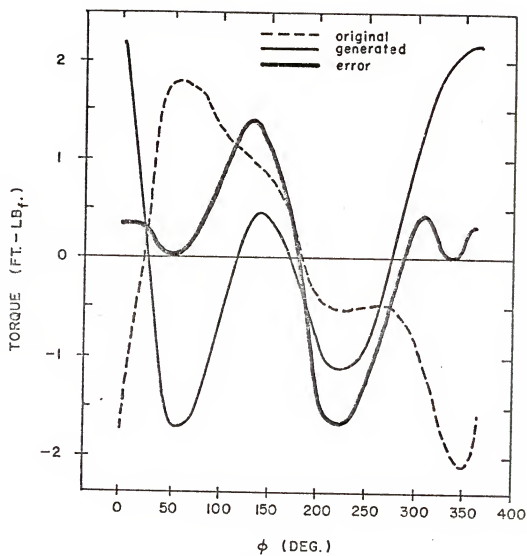


FIGURE VI - 17

satisfied with a given set-up, a co-driver may complain vociferously about the car's handling characteristics. That the number of such complaints is directly proportional to comparative lap times should not be suprising, some racing drivers being nearly as temperamental as their respective machinery. Recently, a rear suspension has been seen on some racing circuits similar to that shown in Fig. VI-18. At first, the assumption is that the unusual complexity is being undertaken in order to separate the spring from the air stream, thus reducing the drag associated with outboard springs. This idea must be discarded since there are much simpler, conventional means of locating the spring inboard. The next assumption is that a drastically non-linear effect was desired. Upon analysis, this too proves to be false, as a plot of torque vs. ϕ for link 1 is essentially linear, shown as a broken line in Fig. VI-19. Hence, the entire apparatus could have been replaced by a torsion bar at pin A. This leaves the final conclusion that the designer has played an excellent game of "cloak and dagger", trying to convince the competition (or even his own drivers) that his efforts have resulted in a superior system.

Regardless of the source of the design, the question is whether or not the complexity may be reduced while still producing the same effects. The possibility of replacement through means of the aforementioned torsion bar will not be considered since the emphasis here is on linear springs.

Considering only the actual suspension geometry, it was desired that the same torque be produced at link 1 as before. The specifications corresponded to the first, central, and last torque values; i.e., the case PP-PP-PP was considered, requiring that the spring constant be unknown.

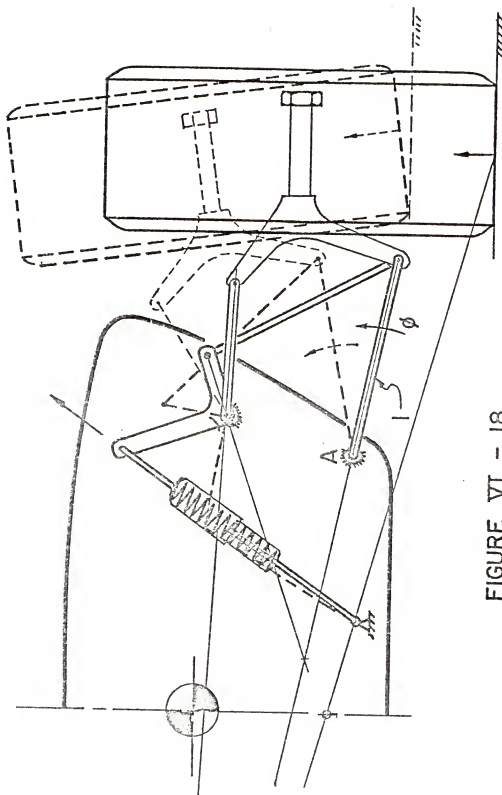


FIGURE VI - 18

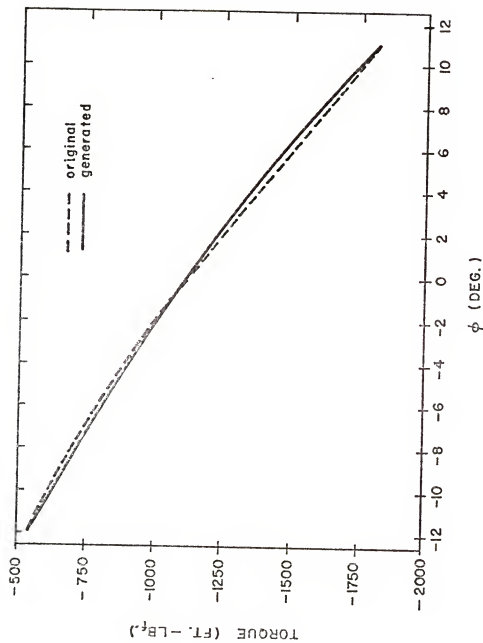


FIGURE VI - 19

Numerous attempts were made to find realistic solutions, but almost all such attempts led to spring anchorage coordinates on the opposite side of the wheels from the frame. A solution which will meet the space limitations is shown in Fig. VI-20, with the resultant torque curve shown as a solid line in Fig. VI-19. The major drawbacks to this solution are the increased size of the hub carrier and the high internal forces in the spring -- on the order of 4000 lbf.

An attempt was next made to attach a spring between the upper and lower A-arms. To accomplish this, it is first necessary to utilize the concepts of angular inversion to "fix" one of the A-arms. Such a configuration is shown in Fig. VI-21. It is then possible to use the original torque specifications in conjunction with an $A_{m\lambda}$ formulation, or to transform the required torque to the hub carrier and use the $T_{m\lambda}$ formulation as described in Chapter V. When attempts were made to find solutions for this problem, it was invariably found that space requirements would not be met; for example, attachment to the lower A-arm would occur at a point below ground level.

Carpet Tufting

Carpet tufting is a process which consists of using a slider-crank mechanism such as shown in Fig. VI-22a to insert yarn through the carpet backing. The yarn is caught in "loopers" and the needle withdrawn, whereupon the cycle is repeated. The entire width of the carpet is tufted at one time, using a long input shaft to drive a number of the slider-crank assemblies. One of the problems encountered on the machine is that the "loopers" miss the tufted yarn inserted through the carpet backing. This in turn is due to the fact that improper phasing occurs within the process. The phasing is in turn distorted by the impact of

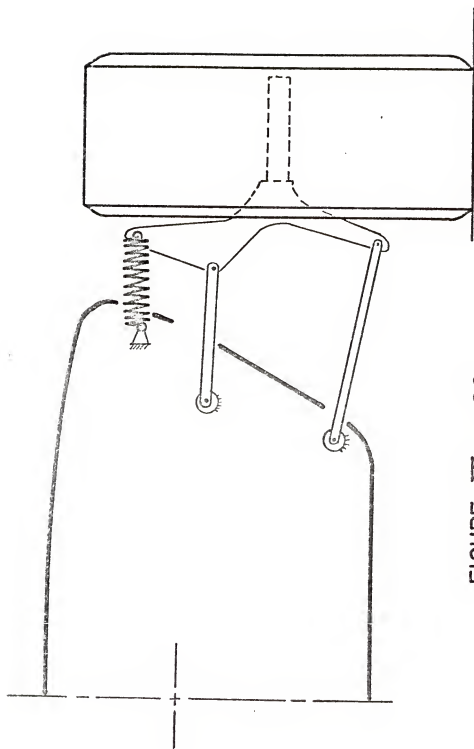


FIGURE VI - 20

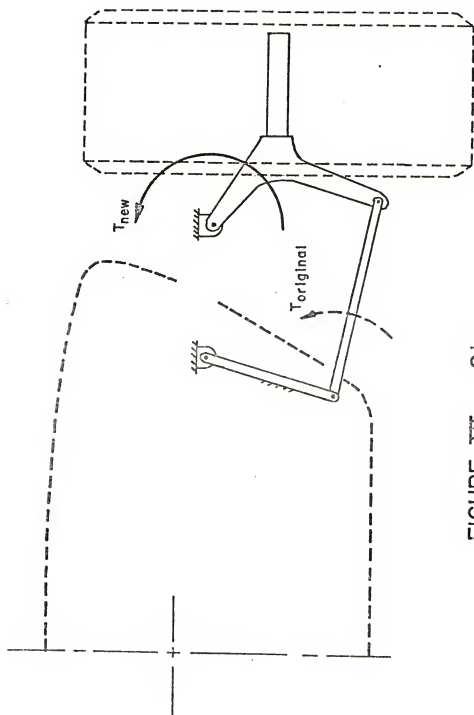


FIGURE VI - 21

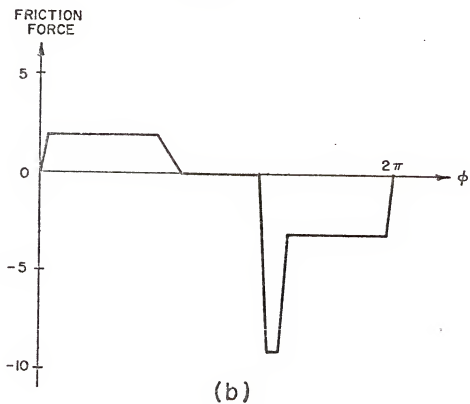
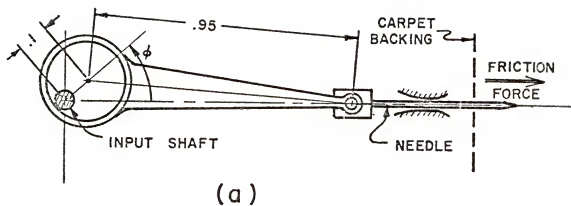


FIGURE VI - 22

the needles when penetrating and withdrawing from the carpet backing. The assumed friction force is shown in Fig. VI-22b. The resulting torque on the input shaft is shown in Fig. VI-23, producing a wind-up condition in the shaft. It should be noted that the motion distortion is not caused by vibration phenomena, but rather by a gross distortion associated with the rotation of the input and directly caused by the induced torque variation.

In order to carry out a balancing process, it should first be noted that the torque is produced by a non-conservative phenomenon. That is, the average torque is not zero as in previous examples, but rather, takes on a negative value as shown in Fig. VI-23. The desired torque to be generated by the insertion of a spring should be the reflection of the original torque about this average. It should also be noted that springs will generally not exhibit the nearly discontinuous behavior desired. This means that a high quality error curve is extremely difficult to achieve. Further, the highly unusual behavior of the desired torque curve means that direct torque specifications are not likely to prove fruitful since choices of particular values of torque are not necessarily "representative" of the curve. Indeed, attempted synthesis using direct torque specifications proved useless. In order to proceed, the desired torque curve is "smoothed" by numerically integrating to arrive at an energy curve. Specifications were then made for 5 FSP in energy with an unknown spring constant. Again, numerous solutions had to be discarded because of space limitations. Of the remaining solutions, a random search was conducted for the one exhibiting the best torque behavior in a "mini-max" sense. The solution is shown in Fig. VI-24a with attendant curves depicted in Fig. VI-25. A second solution was found by specifying

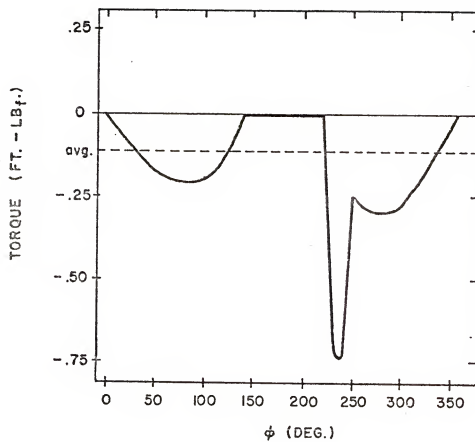
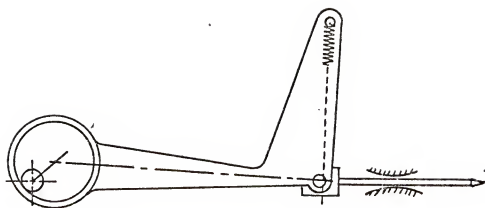
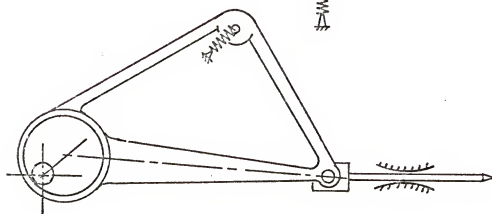


FIGURE VI - 23



(a)



(b)

FIGURE VI - 24

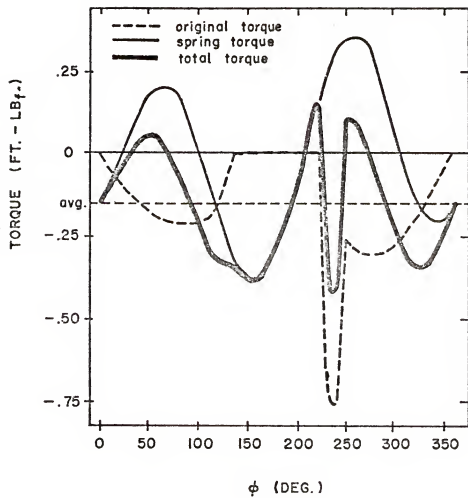


FIGURE VI - 25

4 FSP in energy with a known spring constant ($K = 30 \text{ lb}_f/\text{ft.}$). Again a random search was conducted, this time for the solution best behaved in an RMS sense. The final solution is shown in Fig. VI-24b with associated curves in Fig. VI-26.

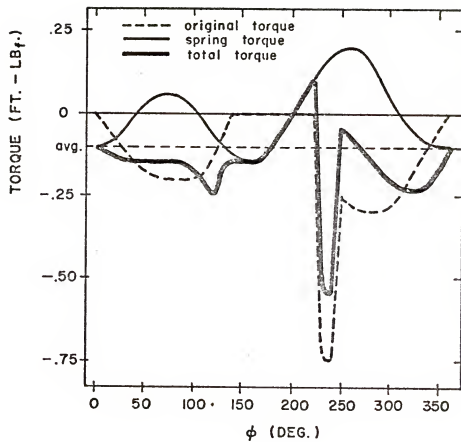


FIGURE VI - 26

CHAPTER VII

CONCLUSIONS

Problem Definition

Torque fluctuations in complex machinery usually result from two distinct phenomena. The most easily recognized sources are those which are position dependent external loads such as stamping operations, gas compression, etc. The second type of fluctuations result from accelerations of the mass or inertia content of machines. The complex geometry associated with many machines produces the non-uniform element velocities even though the input shaft turns at a constant rate. When the balancing techniques of the previous chapters are applied, the resulting spring parameters achieve the required function approximation in a positional sense; that is, the torque generated by the spring at the input is dependent solely upon the input position, and not upon velocity, acceleration, etc. This in turn means that a spring balance for the second type of torque variation is only valid for a given design speed. Since the kinetic energy of a machine, and hence the torque at the input, is proportional to the square of the input speed, the spring balancing may always be "retuned" by altering the spring constant according to the square of the speed ratio $K_{\text{new}} = K_{\text{old}} \left(\frac{\omega_{\text{new}}}{\omega_{\text{old}}} \right)^2$. This in turn means that a total re-synthesis of the spring parameters is unnecessary.

For machine processes which actually produce both types of torque fluctuations, the designer should strongly consider the independent

balancing of the two effects. In such a case, the spring which balances position dependent variations will always accomplish the prescribed goals. For the speed dependent variation, the spring constant may be altered as described above. Otherwise, the single spring which balances both tasks must be completely re-synthesized for the new operating conditions.

Problem Solutions

The body of analytics set forth in the previous chapters has been directed towards the solution of spring-associated parameters applied to non-linear mechanisms. By requiring that certain precision conditions be satisfied, the solution set is known to perform, at least partially, according to the requirements of the designer. If additional quality is required, some optimization processes such as those suggested earlier, may be used to select the "best" solution or combination of solutions.

From the examples shown in Chapter VI, it would appear that high quality results are only obtained when the designer uses an auxiliary mechanism rather than attaching the spring directly to the existing mechanism. Such a conclusion is partially correct in that the designer must have the greatest degree of flexibility when free to choose the driving mechanism. This must be tempered with the realization that the introduction of such a new mechanism adds at least four new uncontrolled parameters to the design process. Further, the examples of Chapter VI which lead to such an impression must be recognized as extremely restrictive; for example, the suspension problem is quite space-limited, while the carpet tufting problem has a particularly "nasty" work function associated with a mechanism which does not have highly non-linear motion --

primarily due to the small relative size of the input crank. Thus, if the designer is forced to choose an auxiliary mechanism, a rather elementary rule of thumb to be borne in mind is "unusual driving torque requirements will demand mechanisms with highly non-linear motion."

The "Pre-Existing Mechanism" Concept

Basically, the history of mechanisms and associated kinematic theory has been directed toward the satisfaction of a functional requirement in the motion sense. That is, the common use of mechanisms is directed towards input-output angular motion generation, point path generation, etc. Even where the mechanism was assumed to pre-exist, such as in the shake balancing process of Berkof and Lowen, the work was confined to the realm of the immediate mechanism -- no attempt was made to use the mechanism to shake balance an aggregate machine.

Here, however, the mechanism is being utilized as a dynamic tool. That essentially the same kinematic body can be extended to accomplish this task is indeed a fortunate circumstance. With the liberation from the historical kinematic concepts, the further extension of the mechanism as a dynamic tool becomes possible in the realm of mass and damping synthesis. Even on the "elementary" level of single mechanism design, such a total analytical development will permit the more exacting design of machines capable of performing tasks with accuracy at high speeds.

Error Analysis

Two basic questions involving error analysis arise immediately. The first involves the actual kinematic analysis of the synthesized parameters to determine behavior over the entire range of the independent variable.

The author has done no computer coding for such analysis, relying instead upon the current effort of Pollock [26]. These analysis programs appear to be quite accurate and easy to use.

The second type of error analysis involves the accuracy with which the synthesized spring parameters may actually be implemented by the practicing designer. Obviously, no attempt will be made to locate, say, the fixed pivot of the spring within $\pm 10^{-6}$ inches. On the other hand, the torque curve to be balanced is probably not known with great precision in the first place. The present work has not attempted to answer these questions, but suggests that such work must be done in order to convince possible industrial users of the acceptability of the process.

Multi-Plane Synthesis

The analytics of Chapters II, III, and IV are directly aimed at finding the parameters associated with the attachment of a single spring from a plane exhibiting general motion to the ground link. The fact that multiple solutions may be utilized does not affect the basic postulation. Consider now the possibility of attaching springs from separated moving planes to the ground link. With no proof given, it is strongly suspected that as many as twelve precision points may be satisfied for springs attached to adjacent links as shown in Fig. VII-1.

Synthesis of Free Length

The most recent efforts indicate that the synthesis of free length is indeed feasible. The controlling energy equation is now

$$\bar{E}_\ell = \frac{1}{2}K(L_\ell - \ell)^2$$

where ℓ is the free length. Rooting both sides and rearranging yields

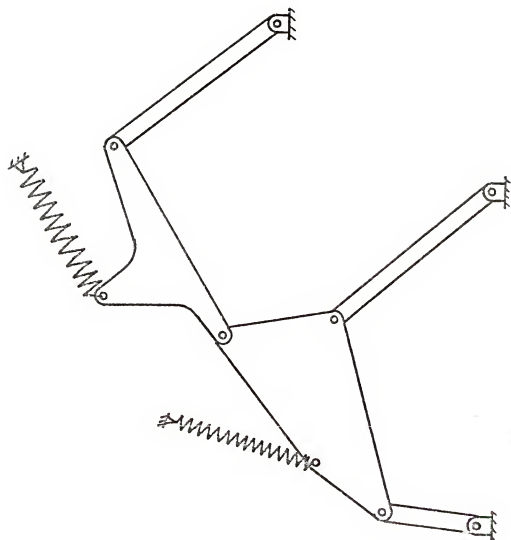


FIGURE VII - I

$$\sqrt{2\delta\bar{E}_\ell} + \mathcal{L} = L_\ell.$$

Squaring both sides gives

$$(\sqrt{2\delta\bar{E}_\ell} + \mathcal{L})^2 = 2\delta\bar{E}_\ell + 2\sqrt{2\delta\bar{E}_\ell}\mathcal{L} + \mathcal{L}^2 = L_\ell^2,$$

and subtracting the reference position gives

$$2\delta(\bar{E}_\ell - \bar{E}_0) + 2\sqrt{2}(\sqrt{\bar{E}_\ell} - \sqrt{\bar{E}_0})\sqrt{\delta}\mathcal{L} = L_\ell^2 - L_0^2. \quad (7-1)$$

The right side of Eq. (7-1) is known as

$$L_\ell^2 - L_0^2 = \sum_{m=0}^6 Z_m A_{m\ell}$$

and, considering the spring constant and energy levels known, the new unknown is $\mathcal{L}$. For the case where the spring constant is unknown, the new unknowns are δ and $\sqrt{\delta}\mathcal{L}$, allowing up to seven-position design. Equation (7-1) is written only for finitely separated positions and it is not yet obvious that the case of multiply separated positions is usable.

APPENDIX A

CHARACTERISTICS OF THE ASYMPTOTE OF CIRCULAR CUBICS

The circular cubic for kinematics or springs may be represented by

$$(u^2 + v^2)(H_1u + H_2v) + H_3u^2 + H_4v^2 + H_5uv + H_6u + H_7v + H_8 = 0. \quad (A-1)$$

In order to find the slope of the asymptote, it is assumed that $v = tu$.

Substitution into the above gives

$$(u^2 + t^2u^2)(H_1u + H_2tu) + \dots = 0.$$

Letting $u \rightarrow \infty$, it is apparent that $H_1u + H_2tu = 0$, so that $t = -\frac{H_1}{H_2}$. Now let the asymptote be represented by the line

$$v = -\frac{H_1}{H_2}u + q. \quad (A-2)$$

Substitution into Eq. (A-1) gives

$$\begin{aligned} & (u^2 + \frac{H_1^2}{H_2^2}u^2 - 2\frac{H_1}{H_2}uq + q^2)(H_2q) + H_3u^2 + \frac{H_1^2H_4}{H_2^2}u^2 - 2\frac{H_1H_4}{H_2}uq \\ & + H_4q^2 - \frac{H_1H_5}{H_2}u^2 + H_5uq + H_6u - \frac{H_1H_7}{H_2}u + H_7q + H_8 = 0, \end{aligned} \quad (A-3)$$

or

$$u^2[(1 + \frac{H_1^2}{H_2^2})H_2q + H_3 + \frac{H_1^2H_4}{H_2^2} - \frac{H_1H_5}{H_2}] + \dots = 0. \quad (A-4)$$

Again letting $u \rightarrow \infty$, the resulting expression for q from Eq. (A-4) is

$$q = - \frac{H_1^2 H_4 + H_2^2 H_3 - H_1 H_2 H_5}{H_2 (H_1^2 + H_2^2)} \quad (A-5)$$

Noting that Eq. (A-3) is a valid general expression and that the coefficient of the u^2 term must be identically zero when the cubic is intersected with the asymptote, the result is

$$\begin{aligned} & (-2 \frac{H_1}{H_2} uq + q^2) (H_2 q) - 2 \frac{H_1 H_4}{H_2} uq + H_4 q^2 + H_5 uq + H_6 u - \frac{H_1 H_7}{H_2} u \\ & + H_7 q + H_8 = 0. \end{aligned}$$

Rearrangement yields

$$\begin{aligned} & -2H_1 H_2 uq^2 + H_2^2 q^3 - 2H_1 H_4 uq + H_2 H_4 q^2 + H_2 H_5 uq + H_2 H_6 u - H_1 H_7 u \\ & + H_2 H_7 q + H_2 H_8 = 0, \end{aligned}$$

and hence

$$u = \frac{H_2 (H_2 q^3 + H_4 q^2 + H_7 q + H_8)}{2H_1 H_2 q^2 + 2H_1 H_4 q - H_2 H_6 + H_1 H_7 - H_2 H_5 q}.$$

The corresponding value of v may be found by a return to Eq. (A-2).

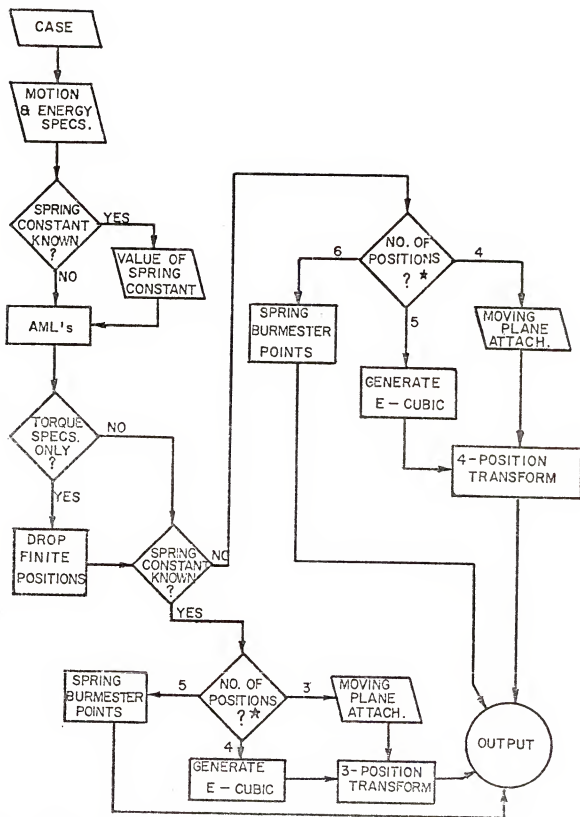
APPENDIX B

COMPUTER CONSIDERATIONS

Because the analytic formulation for the various multiply separated positions is essentially the same, it becomes immediately practical to create a single computer program for all such cases. It is only necessary that the computer be informed of the particular case under consideration so that the $A_{m\ell}$'s (or $T_{m\ell}$'s) are properly created. Further, other minor anomalies exist when considering the spring constant unknown. The essentials of the program flow are shown in Fig. B-1. The entire process is started by typing "MSP", at which point the computer interactively asks the appropriate questions. At the completion of computation, the operator may type "OUTPUT" to have appropriate unknowns displayed.

One problem associated with springs occurs, as mentioned in Chapter II; the fact that the intersection of the E-cubic and its asymptote often does not lie in the "useful" portion of the locus. A flow chart for correcting this problem is depicted in Fig. B-2.

Since the programs have actually been created using $T_{m\ell}$'s, the program is also immediately useful for the partial time state synthesis of mechanisms. This is accomplished by specifying 0 (zero) for all energy levels. Due to a division definition in APL, it is necessary that a non-zero value for the spring constant be entered. Further, since kinematic synthesis is a subset of partial time state synthesis, entry of appropriate values of α , β , γ and their derivatives can be used to find kinematic constraints.



* THE NUMBER OF POSITIONS IS REDUCED BY ONE
IF "DIRECT TORQUE" SPECIFICATIONS ARE IN EFFECT.

FIGURE B - 1

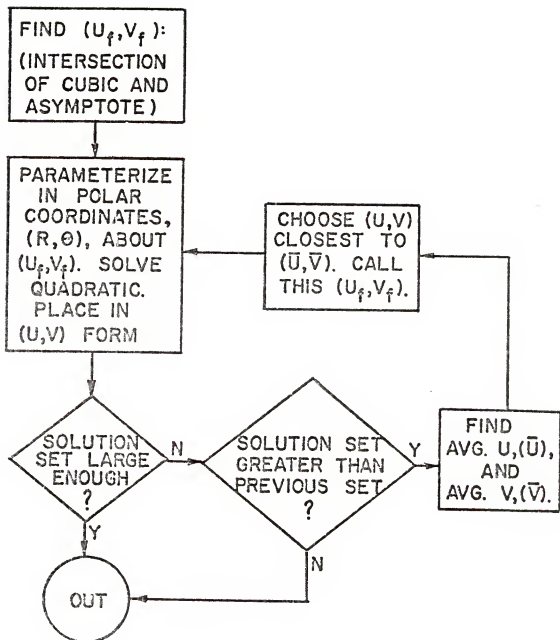


FIGURE B-2

∇ AMLS

- [1] $A1 \leftarrow ((A \times M[L;1]) \times CG) + ((B \times M[L;2]) \times SG) - ((A0 \times M[1;1]) \times CG0 + 1 \times CG + 2 \times OM[L;3]) + (B0 \times M[1;2]) \times SG0 + 1 \times SG + 1 \times OM[(L1+L+(\underline{K}=0)/\rho \underline{K});3]$
- [2] $A2 \leftarrow (-A \times SG) + (B \times CG) + (A0 \times SG0) - B0 \times CG0$
- [3] $\rightarrow 5 \times \text{!} K \leftarrow \text{!} N' = 1 \uparrow \square, \square \leftarrow \text{'SPRING CONSTANT KNOWN?'}$
- [4] $K \leftarrow CTD \square, 0 \uparrow \square \leftarrow \text{'VALUE?'}$
- [5] $AML[L;1] \leftarrow A1, A2, (CG - CG0), (SG - SG0), (A - A0), (B - B0), (0.5 \times - / (A, A0, B, [1.5] B0) \times 2), [1.5] (M[L;4] - M[1;4]) \div K$
- [6] $\rightarrow 0 \times \text{!} \sim 1 \in \underline{K}$
- [7] $AML[L;2] \leftarrow ((SG \times Z\rho SG) \times - (AP \leftarrow, M[L;1]) + (GP \times B + Z\rho B) \rightarrow (CG \times Z\rho CG) \times (BP \leftarrow, M[L;2]) - (A \leftarrow (Z \leftarrow \rho L) \rho A) \times GP \leftarrow, M[(L+(\underline{K}=1)/\rho \underline{K});3]$
- [8] $AML[L;1] \leftarrow (AP \times CG) - (A \times GP \times SG) - (BP \times SG) + B \times GP \times CG$
- [9] $AML[L;2+16] \leftarrow (-GP \times SG), (GP \times CG), AP, BP, ((A \times AP) + B \times BP), [1.5] M[L;4] \div K$
- [10] $\rightarrow 0 \times \text{!} \sim 2 \in \underline{K}$
- [11] $AML[L;1] \leftarrow (CG \times Z\rho CG) \times (2 \times GP \times BP + Z\rho BP) + (APP \leftarrow, M[L;1]) - ((A \leftarrow Z \rho A) \times (GP \times Z\rho GP) \times 2) - (B \leftarrow (Z \leftarrow \rho L) \rho B) \times GPP \leftarrow, M[(L+(\underline{K}=2)/\rho \underline{K});3]$
- [12] $AML[L;1] \leftarrow AML[L;1] + (SG \times Z\rho SG) \times - (2 \times GP \times AP + Z\rho AP) + (A \times GPP) - (BPP \leftarrow, M[L;2]) - B \times GP \times 2$
- [13] $AML[L;2] \leftarrow (SG \times -APP - (A \times GP \times 2) - (2 \times BP \times GP) + B \times GPP) + CG \times BPP - (2 \times AP \times GP) + (A \times GPP) + B \times GP \times 4$
- [14] $AML[L;2+16] \leftarrow (-GPP \times SG) + CG \times GP \times 2, ((GPP \times CG) - SG \times GP \times 2), APP, BPP, ((AP \times 2) + (APP \times A) + (BP \times 2) + B \times BPP), [1.5] M[L;4] \div K$
- [15] $\rightarrow 0 \times \text{!} \sim 3 \in \underline{K}$
- [16] $AML[L;2] \leftarrow (A \times GP \times 3) - (3 \times (AP \leftarrow Z\rho AP) \times GPP + Z\rho GPP) + (3 \times (APP \leftarrow Z\rho APP) \times GP \times Z\rho GP) + (A \leftarrow (A \leftarrow \rho L) \rho A) \times G3 \leftarrow, M[(L+(\underline{K}=3)/\rho \underline{K});3]$
- [17] $AML[L;2] \leftarrow (CG \times Z\rho CG) \times AML[L;2] - (3 \times GP \times GPP \times B + Z\rho B) + (3 \times GP \times GP \times BP + Z\rho BP) - B3 \leftarrow, M[L;2]$
- [18] $AML[L;2] \leftarrow AML[L;2] + (SG \times Z\rho SG) \times (3 \times A \times GP \times GPP) + (3 \times AP \times GP \times GP) + (B \times GP \times 3) - A3 \leftarrow, M[L;1]$
- [19] $AML[L;2] \leftarrow AML[L;2] - SG \times (B \times G3) + 3 \times (BP \times GPP) + GP \times BPP + Z\rho BPP$
- [20] $AML[L;1] \leftarrow SG \times B3 + (A \times GP \times 3) - (A \times G3) + 3 \times (A \times GPP) + (APP \times GP) + (B \times GP \times GPP) + BP \times GP \times GP$
- [21] $AML[L;1] \leftarrow AML[L;1] + CG \times A3 + (B \times G3) - (B \times GP \times 3) + 3 \times (A \times GP \times GPP) + (AP \times GP \times GP) - (BP \times GPP) + BPP \times GP$
- [22] $AML[L;2+16] \leftarrow ((SG \times -G3 - GP \times 3) - CG \times 3 \times GP \times GPP), ((-SG \times 3 \times GP \times GPP) + CG \times G3 - GP \times 3), A3, B3, ((A \times A3) + (B \times B3) + 3 \times (AP \times APP) + BP \times BPP), M[L;4] \div K$

∇

∇ BURM5 A;A;B;C;D;LA;LB;LC;LD;LE;LF;AF;AD;BD;AE;BE;CE;AC;AF;BF;CE;CF;CD

- [1] A FINDS BURMESTER SOLUTIONS FOR FIVE POSITIONS.
- [2] $LF \leftarrow 2 \ 1 \ \rho \ 1 \ 1 \times 2 + A + (D \leftarrow - \div \underline{A} \ \underline{A}[; \ 3 \ 4 \ 5 \ 6]) \times (\underline{A} \ \underline{d}[; \ 7 \ 4 \ 5 \ 6]), (\underline{A} \ \underline{A}[; \ 3 \ 7 \ 5 \ 6]), (\underline{A} \ \underline{A}[; \ 3 \ 4 \ 7 \ 6]), \underline{A} \ \underline{A}[; \ 3 \ 4 \ 5 \ 7]$

```

[3] LD+ 2 1 p 1- 1 ×2+B+D×(Δ A[; 1 4 5 6]),(Δ A[;
    3 1 5 6]),(Δ A[; 3 4 1 6]),Δ A[; 3 4 5 1]
[4] LA+ 2 1 p φ2+C+D×(Δ A[; 2 4 5 6]),(Δ A[; 3 2
    5 6]),(Δ A[; 3 4 2 6]),Δ A[; 3 4 5 2]
[5] COM
    V

```

▽ BURM6 A;A;B;C;D;LA;LB;LC;LD;LE;LF;AB;AC;AD;AE;AF;BC;
BD;BE;CD;CE;CF;BF

```

[1] D+÷Δ A[; 3 4 5 6 8]
[2] LF+ 2 1 p- 1 1 ×2+A+D×(Δ A[; 7 4 5 6 8]),(Δ A[;
    3 7 5 6 8]),(Δ A[; 3 4 7 6 8]),(Δ A[; 3 4 5
    7 8]),Δ A[;2+;5]
[3] LD+ 2 1 p 1- 1 ×(B+D×(Δ A[; 1 4 5 6 8]),(Δ A[;
    3 1 5 6 8]),(Δ A[; 3 4 1 6 8]),(Δ A[; 3 4 5
    1 8]),Δ A[; 3 4 5 6 1])[3 4]
[4] LA+ 2 1 p(C+D×(Δ A[; 2 4 5 6 8]),(Δ A[; 3 2
    5 6 8]),(Δ A[; 3 4 2 6 8]),[E A[; 3 4 5 2 8]),Δ A[;
    3 4 5 6 2])[4 3]
[5] COM
[6] K←(A+K>0)/K+÷A[5]+(B[5]×U)+C[5]×V
[7] U←A/U
[8] V←A/V
[9] US←A/US
[10] VS←A/VS
[11] U1←A/U1
[12] V1←A/V1

```

▽

▽ Z←CASE K;N

```

[1] K← 1 2[1+ 'P'≠K],Z←10
[2] Z←Z, 1+1-1+N+K;2
[3] →2×0<pK+N+K

```

▽

▽ CGM

```

[1] LB+ 2 1 p(C[3]+B[4]),B[3]-C[4]
[2] LC+ 2 1 p(A[4]-C[1]),A[3]+C[2]
[3] LE+ 2 1 p(A[3]-B[1]),B[2]-A[4]
[4] E←((AD*2)-AB×BD),(2×(AD+Δ LA,LD)×AE+Δ LA,LE)-((AB+Δ
    LA,LB)×(BE+Δ LB,LE)+CD+Δ LC,LD)+(AC+Δ LA,LC)×BD+Δ LB,
    LD
[5] E←E,(AE*2)+(2×AD×AF+Δ LA,LF)-(AD×(BF+Δ LB,LF)+CE+Δ LC
    ,LE)+AC×BE+CD
[6] E←E,((2×AE×AF)-(AC×BF+CE)+AB×CF),(AF*2)-AC×CF+Δ LC,LF
[7] US←-A[3]+(B[3]×U)+C[3]×V←-((AD×U*2)+(AE×U)+AF)÷AC+AB×
    U←(0=U[;2])/,(U+QUART E)[;1]
[8] VS←-A[4]+(B[4]×U)+C[4]×V
[9] U1←(U×2OM[1;3])-(V×1OM[1;3])-M[1;1]
[10] V1←(U×1OM[1;3])+(V×2OM[1;3])+N[1;2]

```

▽

∇ Z+CTD Y;P;X;K;D
 [1] $\rightarrow 8 \times 1 \text{'OUT' A.} = 3pY$
 [2] $Z+0+Y+^{-1}+0123456789.^{-1} \text{' } Y$
 [3] $\rightarrow (0=K+1+(12 \neq Y)/1p, Y)/0$
 [4] $K+(Y+(K-1)+Y) \text{' } 12$
 [5] $P+(X+(K-1)+Y) \text{' } 10$
 [6] $Z+Z, (^{-1} * D) \times (10 \text{' } (D+11 \in X)+X+(P-1)+X)+(10 \text{' } D) \div 10 * p D+P+X$
 [7] $\rightarrow 3 \times 0 < p Y+K+Y$
 [8] $\rightarrow$
 ∇

∇ Z+CUBIC X;A;B;C;D
 [1] $\rightarrow 4 \times 1 (Z=0) \vee 0 \leq X-((Z+(A \times C)-B * 2) * 3)+(D+(-B * 3)+0.5 \times (3 \times A \times (B+X[2] \div 3) \times C+X[3] \div 3)-A \times (A+1+X) \times D+^{-1}+X) * 2$
 [2] $Z+(3 \text{' } 1 \text{' } p((2 \times \times D) \times (Z+0.5) \times 2 \circ ((\div 3) \times ^{-2} \circ (|D) \div (Z+^{-}Z) * 1.5)+0 \text{' } 1 \text{' } 2 \times 02 \div 3) \text{' } , 0$
 [3] $\rightarrow 0, , Z+(Z-3 \text{' } 2 \text{' } p B, 0) \div A$
 [4] $Z+3 \text{' } 2 \text{' } p(\div A) \times (-B+ / C), 0, Z, D, (Z+(-(\div 2) \times + / C)-B), -D+(\div 2) \times (3 \times 0.5) \times - / C+(D+X, -X+X \times 0.5) * \div 3$
 ∇

∇ A KINSRING H;A;B;C;G;N;Y;Z;X
 [1] $\text{A FINDS THE INTERSECTION OF THE KINEMATIC CUBIC AND THE E-CUBIC.}$
 [2] $G+1+M[;4]-M[1;4]$
 [3] $A+((\Delta G, \underline{A}[; 3 \text{' } 6])+\Delta G, \underline{A}[; 5 \text{' } 4]) \div C+\Delta G, \underline{A}[; 3 \text{' } 4]$
 [4] $B+((\Delta G, \underline{A}[; 6 \text{' } 4])+\Delta G, \underline{A}[; 5 \text{' } 3]) \div C$
 [5] $C+(\Delta G, \underline{A}[; 5 \text{' } 6]) \div C$
 [6] $Y+0 \text{' } 0, -0.5 \times ((3 \times H[1] \times A)+H[2] \times B), (H[1] \times A)+3 \times H[2] \times B$
 [7] $Y+Y, -(H[1] \times B)+H[2] \times A), -(H[3] \times A)+(H[5] \times B \div 2)-0.25 \times (3 \times H[1] \times A * 2)+(2 \times H[2] \times A \times B)+H[1] \times B * 2$
 [8] $Y+Y, -(H[4] \times B)+(H[5] \times A \div 2)-0.25 \times (2 \times H[1] \times A \times B)+(3 \times H[2] \times B * 2)+H[2] \times A * 2$
 [9] $Z+H+Y+Y, (0.25 \times (H[3] \times A * 2)+(H[4] \times B * 2)+(H[5] \times A \times B)-(2 \times H[6] \times A)+2 \times H[7] \times B)-0.125 \times ((A * 2)+B * 2) \times (H[1] \times A)+H[2] \times B$
 [10] $C+C-0.25 \times (A * 2)+B * 2$
 [11] $N+(Z[4]-Z[3]), (Z[7]-C \times Z[2]), (Z[8]-C \times Z[3]), \underline{A}[5], Z[6]-C \times Z[1]$
 [12] $Y+QUART(N[1 \text{' } 4]+. * 2), (2 \times N[1 \text{' } 4]+. \times N[2 \text{' } 5]), ((N[1] \times 2), 1, C, 1)+. \times N[3], N[2 \text{' } 4 \text{' } 5] * 2), (2 \times + / (1, C) \times N[2 \text{' } 4] \times N[3 \text{' } 5]), (1, C)+. \times N[3 \text{' } 5] * 2$
 [13] $X+(((- / Z[; 4 \text{' } 3]) \times Y * 2)+(Y \times - / Z[; 7 \text{' } 2] \times C)+- / Z[; 8 \text{' } 3] \times C) \div - / (Z+((p Y), 8) p Z[; 6 \text{' } 1 \text{' } 5] \times (C+((p Y), 2) p 1, C), Y+(Y[; 2]=0) / Y[; 1]$
 [14] $U1+((U+X-A \div 2) \times A+2 \circ N[1; 3])-((V+Y-B \div 2) \times B+1 \circ N[1; 3]) -M[1; 1]$

[15] $V1 \leftarrow (U \times B) + (V \times A) + M[1; 2]$

[16] $MSP3$

▽

▽ $KUBIC\ A; P; A1; UF; VF; B1; C1; SP; CP; BOA; R; Q; T; UT; VT; C; RU; B$

[1] $\#$ FINDS POINTS ON THE E-CUBIC.

[2] $\rightarrow 4 \times 1; \underline{L} = 1$

[3] $\rightarrow 5, H \leftarrow MAKEH\ A[; 16], +/A[; 7\ 8]$

[4] $H \leftarrow (A[1; 8] \times MAKEH\ 1\ 0\ +A) - (A[2; 8] \times MAKEH\ A[1\ 3\ 4;])$
 $- (A[3; 8] \times MAKEH\ A[1\ 2\ 4;])$ $- A[4; 8] \times MAKEH\ A[1\ 2\ 3;]$

[5] $Q \leftarrow ((\times/H[1\ 2\ 5]) - (\times/H[1\ 1\ 4]) + \times/H[2\ 2\ 3]) \div H[2] \times (H[1] \times 2) + H[2] \times 0; UT \leftarrow VT \leftarrow 2$

[6] $VF \leftarrow Q + T \times UF \leftarrow ((H[4] \times Q \times 2) + (H[7] \times Q) + H[8] + H[2] \times Q \times 3) \div (-2 \times H[1] \times Q \times 2) + (2 \times H[4] \times T \times Q) + (H[5] \times Q) + H[6] + H[7] \times T \leftarrow -H[1] \div H[2]$

[7] $A1 \leftarrow (H[1] \times CP + 2 \times P) + H[2] \times SP + 1 \times P \leftarrow (-5 + 5 \times 136) \times 0 \div 180$

[8] $B \leftarrow (H[3] \times CP \times CP) + (H[4] \times SP \times SP) + (H[5] \times SP \times CP) + (C1 + (H[1] \times UF) + H[2] \times VF) + (B1 + 2 \times (UF \times CP) + VF \times SP) \times A1$

[9] $C \leftarrow (H[6] \times CP) + (H[7] \times SP) + (B1 \times C1) + ((UF \times 2) + VF \times 2) \times A1 + (2 \times H[3] \times UF \times CP) + (2 \times H[4] \times VF \times SP) + H[5] \times (VF \times CP) + UF \times SP$

[10] $R \leftarrow BOA + B1 + (C1 + ((BOA - B \div 2 \times A1) \times 2) - C \div A1) \times 0.5$

[11] $R \leftarrow (B1 + 72 \times 0 \leq C1) / R + R, BOA \leftarrow B1$

[12] $U \leftarrow UF + R \times B1 / 72 \times CP$

[13] $V \leftarrow VF + R \times B1 / 72 \times SP$

[14] $\rightarrow LOOP \times (RU > \rho \times UT) \wedge 40 > RU + \rho \times U$

[15] $LOOP: \rightarrow LOOP + 3 \times 12 = \rho \times U$

[16] $B \leftarrow (+/1 + \sqrt{1 + U[C + H[U]}) \div RU - 2$

[17] $\rightarrow 20, C \leftarrow (+/1 + \sqrt{1 + V[C]}) \div RU - 2$

[18] $B \leftarrow (+/U) \div RU + \rho \times U$

[19] $C \leftarrow (+/V) \div RU$

[20] $UF \leftarrow (UT + U) [R + R1] / R + ((U - B) \times 2) + (V - C) \times 2]$

[21] $\rightarrow 8, VF \leftarrow (VT + V) [R]$

▽

▽ $H \leftarrow MAKEH\ A$

[1] $\#$ CREATES THE COEFFICIENTS OF THE CIRCULAR CUBIC.

[2] $H \leftarrow (\underline{\Delta}\ A[; 1\ 3\ 4]), (\underline{\Delta}\ A[; 2\ 3\ 4]), (UF + \underline{\Delta}\ A[; 7\ 3\ 4]) + (\underline{\Delta}\ A[; 1\ 3\ 6]) + \underline{\Delta}\ A[; 1\ 5\ 4]$

[3] $H \leftarrow H, (UF + (\underline{\Delta}\ A[; 2\ 5\ 3]) + \underline{\Delta}\ A[; 2\ 6\ 4]), (\underline{\Delta}\ A[; 1\ 6\ 4]) + (\underline{\Delta}\ A[; 2\ 3\ 6]) + (\underline{\Delta}\ A[; 2\ 5\ 4]) + \underline{\Delta}\ A[; 1\ 5\ 3]$

[4] $H \leftarrow H, ((\underline{\Delta}\ A[; 7\ 3\ 6]) + (\underline{\Delta}\ A[; 1\ 5\ 6]) + \underline{\Delta}\ A[; 7\ 5\ 4]), ((\underline{\Delta}\ A[; 7\ 6\ 4]) + (\underline{\Delta}\ A[; 2\ 5\ 6]) + \underline{\Delta}\ A[; 7\ 5\ 3]), \underline{\Delta}\ A[; 7\ 5\ 6]$

▽

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▽ MSP;K;N;NN;A1;A2;SG;CG;SGO;CGO;A;B;AO;BO;AP;BP;GP;APP
;BPP;GPP;L;L;L1;L2
[1] M← 0 4 ρ AML←((ρK),8)ρN+1↑K-K+CASE□,0↑□+'WHAT CASE? (
E.G., PP-PP)'
[2] NN← 13 2 ρ'0 1 2 3 4 5 6 7 8 9 101112'
[3] →(3+N≥ρK),M←M,[1](1,1,(0+180),1)×□,0↑□+'ALPHA, BETA,
GAMMA, ENERGY FOR L = ',(,NN[N;1]),', K = ',L2+,NN[1+
K[N+N+1-;]
[4] AMLS
[5] AML← 1 0 +AML
[6] →GO×\N'=1↑□,□+'IS THE DATA SPECIFIED FOR TORQUE ONLY
?'
[7] AML←(∼1+(1ρK)εL1)≠AML
[8] GO→((L× 3 4 5 =1ρAML),(∼L'× 2 3 4 =1ρAML)/FOUR,FIVE
,SIX,K3,K4,K5
[9] K3:U+CTD□,□+'U?'
[10] V←CTD□,□+'V?'
[11] MSP3
[12] →0
[13] K4:KUBIC AML
[14] MSP3
[15] →0
[16] K5:BURM5 AML[;16],+/AML[; 7 8]
[17] →0
[18] FOUR:U+CTD□,□+'U?'
[19] V←CTD□,□+'V?'
[20] MSP4 AML
[21] →0
[22] FIVE:KUBIC AML
[23] MSP4 AML
[24] →0
[25] SIX:BURM6 AML
▽
▽ MSP3;D1;D2;E1;E4;F1;F2;Δ
[1] A THREE POSITION TRANSFORM FROM S(U,V) TO O(US,VS).
[2] D1←AML[1;8]+AML[1;7]+(U×AML[1;1])+V×AML[1;2]
[3] D2←AML[2;8]+AML[2;7]+(U×AML[2;1])+V×AML[2;2]
[4] E1←AML[1;5]+(U×AML[1;3])-V×AML[1;4]
[5] E2←AML[2;5]+(U×AML[2;3])-V×AML[2;4]
[6] F1←AML[1;6]+(U×AML[1;4])+V×AML[1;3]
[7] F2←AML[2;6]+(U×AML[2;4])+V×AML[2;3]
[8] US←((D1×F2)-D2×F1)+Δ+(E1×F2)-E2×F1
[9] VS←((E1×D2)-E2×D1)*Δ
[10] U1←(U×2OM[1;3])-(V×1OM[1;3])-M[1;1]
[11] V1←(U×1OM[1;3])+(V×2OM[1;3])+M[1;2]

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▽

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V MSP4 A;D1;D2;D3;E1;E2;E3;F1;F2;F3;B;C;G;G;Δ
[1] A FOUR POSITION TRANSFORM FROM S(U,V) TO O(US,VS) A
ND K
[2] D1+A[1;7]+(U×A[1;1])+V×A[1;2]
[3] D2+A[2;7]+(U×A[2;1])+V×A[2;2]
[4] D3+A[3;7]+(U×A[3;1])+V×A[3;2]
[5] E1+A[1;5]+(U×A[1;3])-V×A[1;4]
[6] E2+A[2;5]+(U×A[2;3])-V×A[2;4]
[7] E3+A[3;5]+(U×A[3;3])-V×A[3;4]
[8] F1+A[1;6]+(U×A[1;4])+V×A[1;3]
[9] F2+A[2;6]+(U×A[2;4])+V×A[2;3]
[10] F3+A[3;6]+(U×A[3;4])+V×A[3;3]
[11] K+(G+K>0)/K+÷((D1×A)-(D2×B)-D3×C)÷Δ+-(G[1]×A+(E2×F3)-
E3×F2)-(G[2]×B+(E1×F3)-E3×F1)-(G+A[;8])[3]×C+(E1×F2)-
E2×F1
[12] →O×1(ρ10)=ρK
[13] US+G/(-(G[1]×(D2×F3)-D3×F2)-(G[2]×(D1×F3)-D3×F1)-G[
3]×(D1×F2)-D2×F1)÷Δ
[14] VS+G/(-(G[1]×(E2×D3)-E3×D2)-(G[2]×(E1×D3)-E3×D1)-G[
3]×(E1×D2)-E2×D1)÷Δ
[15] U1+((U+G/U)×2OM[1;3])-((V+G/V)×1OM[1;3])-M[1;1]
[16] V1+(U×1OM[1;3])+(V×2OM[1;3])+M[1;2]
[17] →0
[18] O:'NEGATIVE SPRING CONSTANT(S).'
```

V

V OUTPUT;Z

```

[1] 'WOULD YOU LIKE LISTED:'
[2] Z+2ρ'Y'=1+□,□+'MOVING PIVOTS (IN MOVING REFERENCE)?'
[3] Z+Z,2ρ'Y'=1+□,□+'MOVING PIVOTS (IN FIXED REFERENCE)?'
[4] Z+Z,2ρ'Y'=1+□,□+'FIXED PIVOTS?'
[5] →7×11=ρK
[6] Z+Z,2ρ'Y'=1+□,□+'SPRING CONSTANTS?'
[7] (7+Z)/U,V,U1,V1,US,VS,[1,5](ρU)ρK
```

V

V Z+QUART X;Y;R;A;B;D;E

```

[1] Y+,1+,CUBIC 1,(-X[2]),((×/X[1 3])-4×X[4]),(-×/X[
1 1 4]+(×/4,X[2 4])-×/(X+1×X[1])[3 3]
[2] →4×1(RA=0),O+A+((X[1]×X[1]×3+4)-2×X[2]),O)-R QP R+(
S>0)φO,(|S+Y-X[2]-X[1]×X[1]÷4)*
0.5
[3] →5,D+(÷2)×2+,2 QR A+B+((×/4,X[1 2])- (8×X[3])+X[1]*
3) QD 4×R
[4] D+(÷2)×2+,2 QR A+B+2×(Y>0)φO,(|Y+(Y×2)-4×X[4])*
0.5
[5] E+(÷2)×2+,2 QR A-B
[6] Z+ 4 2 ρ(A+R+D),(A+R-D),(A-R-E),(A+(-X[1]÷4),O)- (R+R÷
2)+E
```

V

$\nabla Z \leftarrow H \text{ } \underline{C} D \text{ } D$
 [1] $Z \leftarrow ((2 \div N, 0) \cdot \underline{C} P \text{ } D[1], -D[2]) \div, 1 \div D \text{ } \underline{C} P \text{ } D[1], -D[2]$
 ∇

$\nabla Z \leftarrow X \text{ } \underline{C} P \text{ } Y$
 [1] $Z \leftarrow (-/X \times Y), +/X \times \phi Y$
 ∇

$\nabla Z \leftarrow H \text{ } \underline{C} R \text{ } X; T; R$
 [1] $T \leftarrow (O2) | X[2] \text{ } V A T A N \text{ } X[1]$
 [2] $R \leftarrow (+/X \times 2) \times 0.5$
 [3] $Z \leftarrow (R \div N) \times Q(2, N) \rho(2 O T), 1 O T \leftarrow (\div N) \times T + (O2) \times^{-1} + \div N$
 ∇

$\nabla Z \leftarrow \Delta \text{ } B; N$
 [1] $Z \leftarrow 1$
 [2] $\rightarrow (1 = \rho, B) / 8$
 [3] $\rightarrow ((0 \text{ } 1 \times \rho B) = N \div^{-1} + (0 = B[; 1]) \div 0) / \text{ } 6 \text{ } 9$
 [4] $B[1 + N, 0;] \leftarrow B[1 + 0, N;]$
 [5] $B[1;] \leftarrow -B[1;]$
 [6] $B \leftarrow (1 \text{ } 1 \div B) - (1 \div B[; 1]) \circ. \times 1 \div B[1;] \div N \div 1 \div, B$
 [7] $\rightarrow 2, Z \leftarrow Z \times N$
 [8] $\rightarrow 0 \times \rho Z \leftarrow Z \times, B$
 [9] $\rightarrow Z \leftarrow 0$
 ∇

APPENDIX C

LISTING OF $T_{m\ell}$ 'S

The following is a tabulation of the $T_{m\ell}$'s. It should be noted that k is the derivative counter and, except where noted, the various derivatives of α , β , γ are evaluated at the ℓ^{th} finite position. Further, the "dot" notation, usually reserved for denoting derivatives with respect to time, is used to denote derivatives with respect to a dummy parameter τ . For kinematic synthesis, $\tau = \gamma$; for partial time state synthesis, $\tau = \text{time}$; and for spring synthesis, $\tau = \phi$, where ϕ is the mechanism angular input.

k	$T_{0\ell}$	$T_{1\ell}$	$T_{2\ell}$	$T_{3\ell}$
0	$\frac{1}{2}[(\alpha_\ell^2 + \beta_\ell^2) - (\alpha_0^2 + \beta_0^2)]$	$(\alpha_\ell \cos \gamma_\ell + \beta_\ell \sin \gamma_\ell) - (\alpha_0 \cos \gamma_0 + \beta_0 \sin \gamma_0)$	$(-\alpha_\ell \sin \gamma_\ell + \beta_\ell \cos \gamma_\ell) - (-\alpha_0 \sin \gamma_0 + \beta_0 \cos \gamma_0)$	$\cos \gamma_\ell - \cos \gamma_0$
1	$\alpha \dot{\alpha} + \beta \dot{\beta}$	$(-\alpha \dot{\gamma} + \dot{\beta}) \sin \gamma + (\dot{\alpha} + \beta \dot{\gamma}) \cos \gamma$	$(-\dot{\alpha} - \beta \dot{\gamma}) \sin \gamma + (-\alpha \dot{\gamma} + \dot{\beta}) \cos \gamma$	$-\dot{\gamma} \sin \gamma$
2	$\dot{\alpha}^2 + \alpha \ddot{\alpha} + \dot{\beta}^2 + \beta \ddot{\beta}$	$(\alpha \ddot{\gamma} - 2\dot{\alpha} \dot{\gamma} - \beta \dot{\gamma}^2 + \ddot{\beta}) \sin \gamma + (-\alpha \dot{\gamma}^2 + \ddot{\alpha} + \beta \ddot{\gamma} + 2\dot{\beta} \dot{\gamma}) \cos \gamma$	$(\alpha \dot{\gamma}^2 - \ddot{\alpha} - \beta \dot{\gamma} - 2\dot{\beta} \dot{\gamma}) \sin \gamma + (-\alpha \ddot{\gamma} - 2\dot{\alpha} \dot{\gamma} - \beta \dot{\gamma}^2 + \ddot{\beta}) \cos \gamma$	$-\ddot{\gamma} \sin \gamma - \dot{\gamma}^2 \cos \gamma$
3	$3\dot{\alpha} \ddot{\alpha} + \alpha \dddot{\alpha} + 3\dot{\beta} \ddot{\beta} + \beta \dddot{\beta}$	$(\alpha \dot{\gamma}^3 - \alpha \ddot{\gamma} - 3\dot{\alpha} \dot{\gamma}^2 - 3\ddot{\alpha} \dot{\gamma} - 3\beta \dot{\gamma} \ddot{\gamma} - 3\dot{\beta} \dot{\gamma}^2 + \ddot{\beta}) \sin \gamma + (-3\alpha \dot{\gamma} \ddot{\gamma} - 3\dot{\alpha} \dot{\gamma}^2 + \ddot{\alpha} - \beta \dot{\gamma}^3 + \beta \ddot{\gamma} + 3\dot{\beta} \dot{\gamma} + 3\ddot{\beta} \dot{\gamma}) \cos \gamma$	$(3\alpha \dot{\gamma} \ddot{\gamma} + 3\dot{\alpha} \dot{\gamma}^2 - \ddot{\alpha} + \beta \dot{\gamma}^3 - \beta \ddot{\gamma} - 3\dot{\beta} \dot{\gamma} - 3\ddot{\beta} \dot{\gamma}) \sin \gamma + (\alpha \dot{\gamma}^3 - \alpha \ddot{\gamma} - 3\dot{\alpha} \dot{\gamma}^2 - 3\ddot{\alpha} \dot{\gamma} - 3\beta \dot{\gamma} \ddot{\gamma} - 3\dot{\beta} \dot{\gamma}^2 + \ddot{\beta}) \cos \gamma$	$(\dot{\gamma}^3 - \ddot{\gamma}) \sin \gamma + (-3\dot{\gamma} \ddot{\gamma}) \cos \gamma$
4	$3\ddot{\alpha}^2 + 4\dot{\alpha} \dddot{\alpha} + \alpha \dddot{\ddot{\alpha}} + 3\dot{\beta}^2 + 4\dot{\beta} \ddot{\beta} + \beta \dddot{\beta}$	$(-\alpha \dot{\gamma}^4 + 6\alpha \dot{\gamma}^2 \ddot{\gamma} + 4\dot{\alpha} \dot{\gamma}^3 - 4\ddot{\alpha} \dot{\gamma} - 6\ddot{\alpha} \dot{\gamma} - 4\ddot{\alpha} \dot{\gamma} + \beta \dot{\gamma}^4 - 4\beta \dot{\gamma} \ddot{\gamma} + \ddot{\beta} - 3\beta \dot{\gamma}^2 - 12\dot{\beta} \dot{\gamma} \ddot{\gamma} - 6\ddot{\beta} \dot{\gamma}^2) \sin \gamma + (\alpha \dot{\gamma}^4 - 4\alpha \dot{\gamma} \ddot{\gamma} - 3\alpha \dot{\gamma}^2 - 12\ddot{\alpha} \dot{\gamma} \ddot{\gamma} - 6\ddot{\alpha} \dot{\gamma}^2 + \ddot{\alpha} + \beta \dot{\gamma}^4 - 6\beta \dot{\gamma}^2 \ddot{\gamma} + 4\dot{\beta} \dot{\gamma} - 4\dot{\beta} \dot{\gamma}^3 + 6\ddot{\beta} \dot{\gamma} + 4\ddot{\beta} \dot{\gamma}) \cos \gamma + (-5\ddot{\alpha} \dot{\gamma} - \alpha \ddot{\ddot{\gamma}} + 30\ddot{\alpha} \dot{\gamma}^2 \ddot{\gamma} + 10\ddot{\alpha} \dot{\gamma}^3 - 10\alpha \dot{\gamma} \ddot{\gamma}^2 + 22\alpha \dot{\gamma}^2 \ddot{\gamma} - 10\ddot{\alpha} \dot{\gamma} - 10\ddot{\alpha} \dot{\gamma} - 5\ddot{\alpha} \dot{\gamma} - \alpha \dot{\gamma}^5 + 5\dot{\beta} \dot{\gamma}^4 + \ddot{\beta} + 10\beta \dot{\gamma}^3 \ddot{\gamma} - 20\dot{\beta} \dot{\gamma} \ddot{\gamma} - 10\beta \ddot{\gamma} \ddot{\gamma} - 5\beta \dot{\gamma} \ddot{\gamma} - 15\dot{\beta} \dot{\gamma}^2 - 30\ddot{\beta} \dot{\gamma} \ddot{\gamma} - 10\ddot{\beta} \dot{\gamma}^2) \sin \gamma + (5\dot{\alpha} \dot{\gamma}^4 + 10\alpha \dot{\gamma}^3 \ddot{\gamma} - 20\ddot{\alpha} \dot{\gamma} \ddot{\gamma} + \ddot{\alpha} - 10\alpha \ddot{\gamma} \ddot{\gamma} - 5\alpha \dot{\gamma} \ddot{\gamma} - 15\ddot{\alpha} \dot{\gamma}^2 - 30\ddot{\alpha} \dot{\gamma} \ddot{\gamma} - 10\ddot{\alpha} \dot{\gamma}^2 + 5\dot{\beta} \dot{\gamma}^4 + \ddot{\beta} - 30\dot{\beta} \dot{\gamma}^2 \ddot{\gamma} - 15\beta \dot{\gamma} \ddot{\gamma}^2 - 10\beta \dot{\gamma}^2 \ddot{\gamma} - 10\ddot{\beta} \dot{\gamma}^3 + 10\ddot{\beta} \dot{\gamma} + 10\ddot{\beta} \dot{\gamma} + \beta \dot{\gamma}^5 + 5\ddot{\beta} \dot{\gamma}) \cos \gamma$	$(-\alpha \dot{\gamma}^4 + 4\alpha \dot{\gamma} \ddot{\gamma} + 3\alpha \dot{\gamma}^2 + 12\ddot{\alpha} \dot{\gamma} \ddot{\gamma} + 6\ddot{\alpha} \dot{\gamma}^2 - \ddot{\alpha} - \beta \dot{\gamma}^4 + 6\beta \dot{\gamma}^2 \ddot{\gamma} + 4\dot{\beta} \dot{\gamma}^3 - 4\dot{\beta} \dot{\gamma} - 6\ddot{\beta} \dot{\gamma} - 4\ddot{\beta} \dot{\gamma}) \sin \gamma + (-\alpha \dot{\gamma}^4 + 6\alpha \dot{\gamma}^2 \ddot{\gamma} + 4\dot{\alpha} \dot{\gamma}^3 - 4\ddot{\alpha} \dot{\gamma} - 6\ddot{\alpha} \dot{\gamma} - 4\ddot{\alpha} \dot{\gamma} + \beta \dot{\gamma}^4 - 4\beta \dot{\gamma} \ddot{\gamma} + \ddot{\beta} - 3\beta \dot{\gamma}^2 - 12\dot{\beta} \dot{\gamma} \ddot{\gamma} - 6\ddot{\beta} \dot{\gamma}^2) \cos \gamma + (-5\dot{\alpha} \dot{\gamma}^4 - 10\alpha \dot{\gamma}^3 \ddot{\gamma} + 20\ddot{\alpha} \dot{\gamma} \ddot{\gamma} - 10\alpha \dot{\gamma} \ddot{\gamma} + 5\alpha \dot{\gamma} \ddot{\gamma} + 15\dot{\alpha} \dot{\gamma}^2 - \ddot{\alpha} + 30\ddot{\alpha} \dot{\gamma} \ddot{\gamma} + 10\ddot{\alpha} \dot{\gamma}^2 - 5\dot{\beta} \dot{\gamma}^4 - \ddot{\beta} + 30\beta \dot{\gamma}^2 \ddot{\gamma} + 15\beta \dot{\gamma} \ddot{\gamma}^2 + 10\beta \dot{\gamma}^2 \ddot{\gamma} + 10\ddot{\beta} \dot{\gamma}^3 - 10\ddot{\beta} \dot{\gamma} - 10\ddot{\beta} \dot{\gamma} - 5\ddot{\beta} \dot{\gamma} - \beta \dot{\gamma}^5) \sin \gamma + (-5\dot{\alpha} \dot{\gamma}^4 - \alpha \dot{\gamma}^5 + 30\ddot{\alpha} \dot{\gamma}^2 \ddot{\gamma} - 5\ddot{\alpha} \dot{\gamma} + 15\alpha \dot{\gamma} \ddot{\gamma}^2 + 10\alpha \dot{\gamma}^2 \ddot{\gamma} + 10\ddot{\alpha} \dot{\gamma}^3 - 10\ddot{\alpha} \dot{\gamma} - 10\ddot{\alpha} \dot{\gamma} - \alpha \dot{\gamma}^5 + 5\dot{\beta} \dot{\gamma}^4 + 10\beta \dot{\gamma}^3 \ddot{\gamma} - 20\dot{\beta} \dot{\gamma} \ddot{\gamma} - 10\beta \ddot{\gamma} \ddot{\gamma} - 5\beta \dot{\gamma} \ddot{\gamma} - 15\dot{\beta} \dot{\gamma}^2 - 30\ddot{\beta} \dot{\gamma} \ddot{\gamma} - 10\ddot{\beta} \dot{\gamma}^2 + \ddot{\beta}) \cos \gamma$	$(6\dot{\gamma}^2 \ddot{\gamma} - \ddot{\gamma}) \sin \gamma + (-4\dot{\gamma} \ddot{\gamma} - 3\dot{\gamma}^2 + \dot{\gamma}^4) \cos \gamma$
5	$10\ddot{\alpha} \ddot{\alpha} + 5\dot{\alpha} \dddot{\ddot{\alpha}} + \alpha \dddot{\ddot{\ddot{\alpha}}} + 10\ddot{\beta} \ddot{\beta} + 5\dot{\beta} \dddot{\ddot{\beta}} + \beta \dddot{\ddot{\ddot{\beta}}}$	$(15\dot{\gamma} \ddot{\gamma}^2 + 10\dot{\gamma}^2 \ddot{\gamma} - \ddot{\gamma} - \dot{\gamma}^5) \sin \gamma + (-10\dot{\gamma} \ddot{\gamma} - 5\dot{\gamma} \ddot{\gamma} + 10\dot{\gamma}^3 \ddot{\gamma}) \cos \gamma$	$(15\dot{\gamma} \ddot{\gamma}^2 + 10\dot{\gamma}^2 \ddot{\gamma} - \ddot{\gamma} - \dot{\gamma}^5) \sin \gamma + (-10\dot{\gamma} \ddot{\gamma} - 5\dot{\gamma} \ddot{\gamma} + 10\dot{\gamma}^3 \ddot{\gamma}) \cos \gamma$	

k	$T_{4\ell}$	$T_{5\ell}$	$T_{6\ell}$	$T_{7\ell}$
0	$\sin\gamma_\ell - \sin\gamma_0$	$\alpha_\ell - \alpha_0$	$\beta_\ell - \beta_0$	$(E_\ell - E_0)$
1	$\dot{\gamma} \cos\gamma$	$\dot{\alpha}$	$\dot{\beta}$	T
2	$-\dot{\gamma}^2 \sin\gamma$ $+\ddot{\gamma} \cos\gamma$	$\ddot{\alpha}$	$\ddot{\beta}$	$\dot{T}$
3	$(-3\dot{\gamma}\ddot{\gamma}) \sin\gamma$ $+(-\dot{\gamma}^3 + \ddot{\gamma}) \cos\gamma$	$\ddot{\alpha}$	$\ddot{\beta}$	$\ddot{T}$
4	$(-3\dot{\gamma}^2 - 4\dot{\gamma}\ddot{\gamma}) \sin\gamma$ $+(-6\dot{\gamma}^2\ddot{\gamma} + \ddot{\gamma}^2) \cos\gamma$	$\ddot{\alpha}$	$\ddot{\beta}$	$\ddot{T}$
5	$(-10\ddot{\gamma}\ddot{\gamma} - 5\dot{\gamma}\ddot{\gamma}^2 + 10\dot{\gamma}^3\ddot{\gamma}) \sin\gamma$ $+(-15\dot{\gamma}\ddot{\gamma}^2 - \dot{\gamma}^5 - 10\dot{\gamma}^2\ddot{\gamma} + \ddot{\gamma}) \cos\gamma$	$\ddot{\alpha}$	$\ddot{\beta}$	$\ddot{T}$

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BIOGRAPHICAL SKETCH

Gary Kreider Matthew was born January 21, 1943 at Lebanon, Pennsylvania. After moving to Florida in 1954, he graduated from Clearwater High School in June, 1960. Starting his college work at the University of Florida in September, 1960, five years elapsed between his sophomore and junior years while working in industry. He returned to the University of Florida in September of 1967, receiving a Bachelor of Science in Mechanical Engineering in December, 1970. He immediately pursued the degree of Master of Engineering which was received in March, 1973.

In May, 1966, he married the former Keilah Hall Coon and has one son, John David, age four.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



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Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



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Professor of Mechanical Engineering
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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



E. K. Walsh
Professor of Engineering Sciences

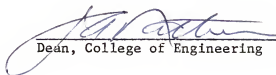
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



R. G. Selfridge
Professor with Computer Information Science

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1975



Dean, College of Engineering

Dean, Graduate School